MATH 285 HOMEWORK 9 SOLUTIONS

Section 9.2

9.

$$a_0 = \int_{-1}^{1} t^2 \, dt = \frac{2}{3}$$

$$a_n = \int_{-1}^{1} t^2 \cos n\pi t \, dt = \frac{-4}{(n\pi)^3} \sin n\pi + \frac{4}{(n\pi)^2} \cos n\pi + \frac{2}{n\pi} \sin n\pi = \frac{4(-1)^n}{(n\pi)^2}$$

$$b_n = \int_{-1}^{1} t^2 \sin n\pi t \, dt = 0$$

We found a_n by integrating by parts twice, and $b_n = 0$ because $t^2 \sin n\pi t$ is an odd function. The Fourier series for $f(t)$ is then

$$f(t) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t$$

11.

$$a_0 = \int_{-1}^{1} \cos \frac{\pi t}{2} \, dt = \frac{2}{\pi} \left[\sin \frac{\pi t}{2} \right]_{-1}^{1} = \frac{4}{\pi}$$

$$a_n = \int_{-1}^{1} \cos \frac{\pi t}{2} \cos n\pi t \, dt = \frac{1}{2} \int_{-1}^{1} [\cos(\frac{\pi t}{2} + n\pi t) + \cos(\frac{\pi t}{2} - n\pi t)] \, dt$$

$$= \frac{1}{2} \left[\frac{1}{\pi(n+1/2)} \sin(\pi(n+1/2)t) + \frac{1}{\pi(n-1/2)} \sin(\pi(n-1/2)t) \right]_{-1}^{1}$$

$$= \left[\frac{1}{\pi(n+1/2)} \sin(\pi(n+1/2)) + \frac{1}{\pi(n-1/2)} \sin(\pi(n-1/2)) \right]$$

$$= \left[\frac{1}{\pi(n+1/2)} (-1)^n + \frac{1}{\pi(n-1/2)} (-1)^{n-1} \right]$$

$$= \frac{(-1)^n}{\pi} \left[\frac{1}{n+1/2} - \frac{1}{n-1/2} \right] = \frac{(-1)^n}{\pi} \frac{-1}{(n+1/2)(n-1/2)} = \frac{(-1)^{n+1}}{\pi(n^2-1/4)} = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \int_{-1}^{1} \cos \frac{\pi t}{2} \sin n\pi t \, dt = 0$$

Where we have $b_n = 0$ because of $\cos \frac{\pi t}{2}$ is an even function. The Fourier series is

$$f(t) \sim \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} \cos n\pi t$$

18. First we derive the Fourier series of the function $f(t)$ of period 2π defined for $0 < t < 2\pi$ by $f(t) = t$. In this, we integrate from 0 to 2π rather than from $-\pi$ to π since we can integrate over any full period.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = \frac{\cos 2n\pi + 2n\pi\sin 2n\pi - 1}{n^2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi} = -\frac{2}{n}$$

$$f(t) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

For $0 < t < 2\pi$,

$$\frac{\pi - t}{2} = \frac{1}{2}[\pi - f(t)] = \sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

Section 9.3

7. For the cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\,dt = \frac{\pi^3}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\cos nt\,dt = -\frac{2[n\pi\cos n\pi + n\pi - 2\sin n\pi]}{n^3\pi} = -\frac{2}{n^2}[1+(-1)^n]$$

Thus the cosine series, valid for $0 < t < \pi$, is

$$t(\pi - t) = \frac{\pi^3}{6} - 4\sum_{n \text{ even}}\frac{\cos nt}{n^2} = \frac{\pi^3}{6} - 4\sum_{k=1}^{\infty}\frac{\cos 2kt}{(2k)^2}$$

For the sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\sin nt\,dt = \frac{2[2 - 2\cos n\pi - 2n\pi\sin n\pi]}{n^3\pi} = \frac{4}{\pi n^3}[1-(-1)^n]$$

Thus the sine series, valid for $0 < t < \pi$, is

$$t(\pi - t) = \frac{8}{\pi}\sum_{n \text{ odd}}\frac{\sin nt}{n^3} = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)t}{(2k-1)^3}$$

9. For the cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin t\,dt = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin t\cos nt\,dt = \frac{2[1 + \cos n\pi]}{\pi(1 - n^2)} = \frac{2[1 + (-1)^n]}{\pi(1 - n^2)}$$

This is only valid if $n > 1$, whereas if $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin t \cos t \, dt = 0$$

Thus the cosine series, valid for $0 < t < \pi$ is

$$\sin t = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos nt$$

For the sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin t \sin nt \, dt = \frac{2\sin n\pi}{\pi(1-n^2)} = 0$$

This is only valid if $n > 1$, whereas if $n = 1$, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 t \, dt = 1$$

Thus the sine series, valid for $0 < t < \pi$ is

$$\sin t = \sin t$$

That is, $\sin t$ is already its sine series.

17. (a) Let f be an even function. Using the substitution $u = -t$, $du = -dt$, we have

$$\int_{-a}^{0} f(t)\, dt = \int_{u=a}^{u=0} f(-u)\,(-du) = -\int_a^0 f(-u)\, du = \int_0^a f(-u)\, du = \int_0^a f(u)\, du = \int_0^a f(t)\, dt$$

(b) Let f be an odd function. Using the same subsitution, and the same steps as before, we have

$$\int_{-a}^{0} f(t)\, dt = \int_{u=a}^{u=0} f(-u)\,(-du) = \int_0^a f(-u)\, du = -\int_0^a f(u)\, dt = -\int_0^a f(t)\, dt$$

19. Begin with the Fourier series, valid for $-\pi < t < \pi$,

$$t = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nt$$

One termwise integration yields

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty} -(-1)^{n+1}\frac{1}{n^2}(\cos nt - 1) = 2\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}\cos nt + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

The constant term $2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is $a_0/2$, where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{t^2}{2}\, dt = \frac{1}{\pi}\int_0^{\pi} t^2\, dt = \frac{1}{\pi}\frac{\pi^3}{3} = \frac{\pi^2}{3}$$

So $a_0/2 = \pi^2/6 = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. (Incidentally, we have shown $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.)

Now we apply termwise integration to

$$\frac{t^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos nt$$

$$\frac{t^3}{6} = \frac{\pi^2 t}{6} + 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^3}\sin nt$$

Now apply termwise integration again to get

$$\frac{t^4}{24} = \frac{\pi^2 t^2}{12} + 2\sum_{n=1}^{\infty} -(-1)^n \frac{1}{n^4}(\cos nt - 1)$$

$$\frac{t^4}{24} = \frac{\pi^2 t^2}{12} - 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^4}\cos nt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

as desired.

20. If we plug in $t = \pi$ into the result of the previous problem, we can use that $\cos n\pi = (-1)^n$ to obtain

$$\frac{\pi^4}{24} = \frac{\pi^4}{12} - 2\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^4}(-1)^n + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$\frac{\pi^4}{24} = \frac{\pi^4}{12} - 2\sum_{n=1}^{\infty}\frac{1}{n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = \frac{\pi^4}{12} - 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^4}$$

$$\frac{\pi^4}{24} = 2\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^4} = 4\sum_{n \text{ odd}}\frac{1}{n^4}$$

Thus

$$\sum_{n \text{ odd}}\frac{1}{n^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

Next we evaluate $S = \sum_{n=1}^{\infty}\frac{1}{n^4}$:

$$S = \sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n \text{ odd}}\frac{1}{n^4} + \sum_{n \text{ even}}\frac{1}{n^4} = \frac{\pi^4}{96} + \sum_{k=1}^{\infty}\frac{1}{(2k)^4} = \frac{\pi^4}{96} + \frac{1}{2^4}\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{96} + \frac{1}{2^4}S$$

This is a linear equation for S that we can solve:

$$S = \frac{16}{15}\frac{\pi^4}{96} = \frac{\pi^4}{90}$$

Lastly we evaluate $T = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$

$$T = \sum_{n \text{ odd}}\frac{1}{n^4} - \sum_{n \text{ even}}\frac{1}{n^4} = 2\left(\sum_{n \text{ odd}}\frac{1}{n^4}\right) - \left(\sum_{n \text{ odd}}\frac{1}{n^4} + \sum_{n \text{ even}}\frac{1}{n^4}\right)$$

$$T = 2\frac{\pi^4}{96} - \frac{\pi^4}{90} = \frac{7\pi^4}{720}$$

Section 9.4

3. (Extra Credit) Let $F(t)$ be the odd function of period 2π such that $F(t) = 2t$ for $0 < t < \pi$. Since $F(t)$ is odd, its Fourier series has only sine terms, and the coefficients are

$$b_n = \frac{2}{\pi} \int_0^{\pi} 2t \sin nt \, dt = \frac{4(-1)^{n+1}}{n}$$

Supposing $x(t)$ to have the form $x(t) = \sum_{n=1}^{\infty} B_n \sin nt$, we obtain

$$\sum_{n=1}^{\infty} -n^2 B_n \sin nt + 3 \sum_{n=1}^{\infty} B_n \sin nt = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n} \sin nt$$

Matching coefficients of $\sin nt$, we get $(3 - n^2)B_n = 4(-1)^{n+1}/n$ so

$$B_n = \frac{4(-1)^{n+1}}{n(3 - n^2)}$$

So the steady periodic solution is

$$x(t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n(3 - n^2)} \sin nt$$