MATH 285 HOMEWORK 8 SOLUTIONS

Section 9.1

13.

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0}(0)\,dt + \int_{0}^{\pi}(1)\,dt\right) = 1$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(0)\cos nt\,dt + \int_{0}^{\pi}(1)\cos nt\,dt\right) = \frac{1}{n\pi}(\sin n\pi - \sin 0) = 0$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(0)\sin nt\,dt + \int_{0}^{\pi}(1)\sin nt\,dt\right) = \frac{-\cos n\pi + \cos 0}{n\pi} = \frac{1-(-1)^n}{n\pi}$$

Thus $b_n = 0$ for even n and $b_n = \frac{2}{n\pi}$ for odd n. The Fourier series for $f(t)$ is

$$f(t) \sim \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi}\sin nt = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right]$$

15.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{1}{\pi}\left[\frac{1}{n^2}\sin nt - \frac{1}{n}t\cos nt\right]_{-\pi}^{\pi} = \frac{1}{\pi}\frac{-1}{n}(2\pi\cos n\pi) = \frac{-2}{n}\cos n\pi$$

Thus $b_n = -2/n$ for n even and $b_n = 2/n$ for n odd. We can also write $b_n = (-1)^{n+1}(2/n)$. The Fourier series is

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nt = 2\left[\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \cdots\right]$$

20.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos nt\,dt = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\sin nt\,dt = 0$$

The value of $\sin \frac{n\pi}{2}$ is 0 if n is even, $+1$ if $n = 1, 5, 9, \dots$, and -1 if $n = 3, 7, 11, \dots$. There are various ways to write the Fourier series, some are

$$f(t) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos nt = \frac{1}{2} + \frac{2}{\pi} \left[\frac{\cos t}{1} - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} + \cdots \right]$$

25. Use the trigonometric identity $\cos^2 x = (1 + \cos 2x)/2$ to obtain

$$f(t) = \cos^2 2t = \frac{1}{2}(1 + \cos 4t) = \frac{1}{2} + \frac{1}{2} \cos 4t$$

This already expresses $f(t)$ as a Fourier series, so we can just match this formula for $f(t)$ with the general form of the Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

To find that $a_0 = 1$, $a_4 = 1/2$, and all other Fourier coefficients are zero.

27. The equation to prove is

$$\int_{\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

Apply the identity $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$ to the integrand.

$$\frac{1}{2} \int_{-\pi}^{\pi} [\cos((m+n)t) + \cos((m-n)t)] \, dt = \frac{1}{2} \left[\frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n} \right]_{-\pi}^{\pi}$$

This is valid as long as the denominators $m + n$ and $m - n$ are not zero. Since m and n are positive integers, $m + n$ is never zero, but $m - n$ can be zero if $m = n$.

In the case where $m \neq n$, we then evaluate at the limits of integration to obtain

$$\frac{1}{2} \left[\frac{\sin((m+n)\pi)}{m+n} + \frac{\sin((m-n)\pi)}{m-n} \right] - \frac{1}{2} \left[\frac{\sin((m+n)(-\pi))}{m+n} + \frac{\sin((m-n)(-\pi))}{m-n} \right]$$

All the terms in this expression are zero because $\sin k\pi = 0$ for any integer k.

In the case where $m = n$, the integral actually becomes

$$\frac{1}{2} \int_{-\pi}^{\pi} [\cos((m+n)t) + 1] \, dt = \frac{1}{2} \left[\frac{\sin((m+n)t)}{m+n} + t \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left[\frac{\sin((m+n)\pi)}{m+n} + \pi \right] - \frac{1}{2} \left[\frac{\sin((m+n)(-\pi))}{m+n} - \pi \right] = \pi$$

28. The equation to prove is

$$\int_{\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

The relevant trigonometric identity is $\sin A \sin B = \frac{1}{2}[-\cos(A+B)+\cos(A-B)]$. Applying this, we get, if we assume $m \neq n$,

$$\frac{1}{2}\int_{-\pi}^{\pi} [-\cos((m+n)t)+\cos((m-n)t)] \, dt = \frac{1}{2}\left[-\frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n}\right]_{-\pi}^{\pi}$$

When we evaluate at the limits, all terms involve $\sin k\pi$, for various integers k, so they are all zero.

If $m = n$, the integral actually becomes

$$\frac{1}{2}\int_{-\pi}^{\pi} [-\cos((m+n)t) + 1] \, dt = \frac{1}{2}\left[-\frac{\sin((m+n)t)}{m+n} + t\right]_{-\pi}^{\pi} = \pi$$

29. The equation to prove is

$$\int_{\pi}^{\pi} \cos mt \sin nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

The relevant trigonometric identity is $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$. Applying this, if we assume $m \neq n$, we obtain

$$\frac{1}{2}\int_{-\pi}^{\pi} [\sin((m+n)t)-\sin((m-n)t)] \, dt = \frac{1}{2}\left[\frac{-\cos((m+n)t)}{m+n} - \frac{-\cos((m-n)t)}{m-n}\right]_{-\pi}^{\pi}$$

Since cos is an even function, when we evaluate at the limits $-\pi$ and π, all terms will cancel, so this is zero. (We could have also seen this by observing that the integrand is an odd function.)

In the case where $m = n$, the result is still zero, but the integrand actually becomes

$$\frac{1}{2}\int_{-\pi}^{\pi} [\sin((m+n)t) - 0] \, dt = 0$$

30. Let $f(t)$ be a piecewise continuous function with period P.
   (a) Let $0 \leq a < P$. We want to show that $\int_P^{a+P} f(t) \, dt = \int_0^a f(t) \, dt$. If we apply the substitution $u = t - P$, $du = dt$ to the first integral, we obtain

$$\int_{t=P}^{t=a+P} f(t) \, dt = \int_{u=0}^{u=a} f(u+P) \, du$$

   Since f is periodic with period P, we have $f(u+P) = f(u)$. Thus

$$\int_{u=0}^{u=a} f(u+P) \, du = \int_{u=0}^{u=a} f(u) \, du$$

   Changing the dummy variable u back to t gives us what we want.

Next, we want to conclude that $\int_a^{a+P} f(t)\,dt = \int_0^P f(t)\,dt$. Divide the interval $[a, a+P]$ into $[a, P]$ and $[P, a+P]$. Then

$$\int_a^{a+P} f(t)\,dt = \int_a^P f(t)\,dt + \int_P^{a+P} f(t)\,dt = \int_a^P f(t)\,dt + \int_0^a f(t)\,dt$$

where we have used what was just proved. But then we see that the integrals over the intervals $[a, P]$ and $[0, a]$ can be combined into an integral over the interval $[0, P]$.

$$\int_a^P f(t)\,dt + \int_0^a f(t)\,dt = \int_0^P f(t)\,dt$$

This completes the proof.

(b) Let A be any number. We want to show that

$$\int_A^{A+P} f(t)\,dt = \int_0^P f(t)\,dt$$

First, find an integer n and a number a with $0 \leq a < P$ such that $A = nP + a$. (It is easy to see that such n and a exist. One can take n to be the integer part of A/P, and then define a accordingly.) Now we apply the substitution $v = t - nP$ to the integral $\int_A^{A+P} f(t)\,dt$:

$$\int_{t=A}^{t=A+P} f(t)\,dt = \int_{v=a}^{v=a+P} f(v+nP)\,dv = \int_{v=a}^{v=a+P} f(v)\,dv = \int_{v=0}^{v=P} f(v)\,dv$$

The first equality is the substitution rule, the second equality uses the fact that f is periodic so $f(v + nP) = f(v)$, and the third equality uses the result of the first part of this problem. Then changing the dummy variable v back to t gives the desired result.