# MATH 285 HOMEWORK 7 SOLUTIONS

## Section 3.5

In all of the problems from section 3.5, there is (mathematical) resonance, and the ordinary undetermined coefficients technique will fail. Therefore we use the annihilator method.

10. This equation may be written $(D^2+9)y = f(x)$, where $f(x) = 2\cos 3x + 3\sin 3x$. An annihilator for $f(x)$ is $A = D^2+9$. Applying this $A$ to both sides of the equation we get $(D^2+9)^2 y = 0$. This is a homogeneous equation with characteristic equation $(r^2+9)^2 = 0$. The roots are $r = \pm 3i$, each with multiplicity two. Our trial solution is the general solution of this latter equation: $y_{\text{trial}}(x) = A\cos 3x + B\sin 3x + Cx\cos 3x + Dx\sin 3x$. Plugging this back into the original equation, we get $(D^2+9)y_{\text{trial}} = C(-6\sin 3x) + D(6\cos 3x)$. Thus we must have $-6C = 3$, $6D = 2$, so $C = -1/2$, $D = 1/3$ ($A$ and $B$ can be anything). Thus a particular solution is $y_p(x) = -(1/2)x\cos 3x + (1/3)x\sin 3x$
14. This equation may be written $(D^4-2D^2+1)y = xe^x$. The differential operator factors as $(D^2-1)^2 = (D-1)^2(D+1)^2$. An annihilator of $xe^x$ is $A = (D-1)^2$. Applying this to both sides of the equation we obtain $(D-1)^4(D+1)^2 y = 0$. Our trial solution is the general solution of this equation $y_{\text{trial}} = Ae^x + Bxe^x + Cx^2e^x + Ex^3e^x + Fe^{-x} + Gxe^{-x}$. Pluggin this back into the original equation, we see that when we compute $(D-1)^2(D+1)^2 y_{\text{trial}}$, the terms with $A, B, F, G$ all go away, since those terms solve the homogeneous equation $(D-1)^2(D+1)^2 y = 0$. So we only need to look at terms with $C$ and $E$. We compute $(D-1)^2(D+1)^2[x^2e^x] = 8e^x$ and $(D-1)^2(D+1)^2[x^3e^x] = 24xe^x + 24e^x$. So we need $8Ce^x + 24E(xe^x + e^x) = xe^x$. So $24E = 1$, and $8C + 24E = 0$. So $E = 1/24$ and $C = -1/8$. Thus a particular solution is $y_p(x) = -(1/8)x^2e^x + (1/24)x^3e^x$.
17. This equation may be written $(D^2+1)y = \sin x + x\cos x$. An annihilator for $\sin x + x\cos x$ is $(D^2+1)^2$. Applying this to both sides of the equation yields $(D^2+1)^3 y = 0$. Our trial solution is therefore $y_{\text{trial}}(x) = A\cos x + B\sin x + Cx\cos x + Ex\sin x + Fx^2\cos x + Gx^2\sin x$. When we compute $(D^2+1)y_{\text{trial}}$, we see that the terms with $A$ and $B$ go away, so we only need to focus on the other four terms. We have $(D^2+1)[x\cos x] = -2\sin x$, $(D^2+1)[x\sin x] = 2\cos x$, $(D^2+1)[x^2\cos x] = -4x\sin x + 2\cos x$, $(D^2+1)[x^2\sin x] = 4x\cos x + 2\sin x$. Thus we need $C(-2\sin x) + E(2\cos x) + F(-4x\sin x + 2\cos x) + G(4x\cos x + 2\sin x) = \sin x +$

$x \cos x$. Thus $-2C + 2G = 1$, $4G = 1$, $2E + 2F = 0$, $-4F = 0$. So $F = 0$, $E = 0$, $G = 1/4$, $C = -1/4$. A particular solution is $y_p(x) = -(1/4)x \cos x + (1/4)x^2 \sin x$.

34. First we find the general solution of $(D^2 + 1)y = \cos x$. The annihilator is $(D^2 + 1)$, so we look at $(D^2 + 1)^2 y = 0$. The trial solution is $y_{\text{trial}} = A \cos x + B \sin x + Cx \cos x + Ex \sin x$. Plugging this into the original equation, we get $(D^2 + 1)y_{\text{trial}} = C(-2 \sin x) + E(2 \cos x)$. Thus we must have $-2C = 0$, and $2E = 1$, while $A$ and $B$ may be anything. So the general solution of the nonhomogenous equation is $y_g(x) = A \cos x + B \sin x + (1/2)x \sin x$. The derivative is $y_g'(x) = -A \sin x + B \cos x + (1/2)(\sin x + x \cos x)$. The initial conditions $y(0) = 1$, $y'(0) = -1$ yield $A = 1$ and $B = -1$. Thus the solution is $y(x) = \cos x - \sin x + (1/2)x \sin x$.

## Section 3.6

2. Trial solution $x_{\text{trial}}(t) = A \sin 3t$ yields $x''_{\text{trial}} + 4x_{\text{trial}} = (-5A) \sin 3t$. For this to equal $5 \sin 3t$ we take $A = -1$. So the particular solution is $x_p(t) = -\sin 3t$. The complementary solution is $x_c(t) = c_1 \cos 2t + c_2 \sin 2t$, and the general solution is $x(t) = c_1 \cos 2t + c_2 \sin 2t - \sin 3t$, with derivative $x'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t - 3 \cos 3t$. Imposing the initial conditions $x(0) = 0, x'(0) = 0$, we get $c_1 = 0$ and $2c_2 - 3 = 0$. Thus $c_2 = 3/2$. The solution is $x(t) = (3/2) \sin 2t - \sin 3t$.
5. Trial solution $x_{\text{trial}}(t) = A \cos \omega t$ gives $(-m\omega^2 + k)A \cos \omega t = F_0 \cos \omega t$. Thus $A = F_0/(k - m\omega^2)$. With $\omega_0 = \sqrt{k/m}$, the general solution is $x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + (F_0/(k - m\omega^2)) \cos \omega t$. Imposing the initial condition $x(0) = x_0$ and $x'(0) = 0$ yields $c_1 + (F_0/(k - m\omega^2) = x_0$, and $\omega_0 c_2 = 0$. Thus $c_2 = 0$ and $c_1 = x_0 - (F_0/(k - m\omega^2))$. The solution is $x(t) = (x_0 - (F_0/(k - m\omega^2))) \cos \omega_0 t + (F_0/(k - m\omega^2)) \cos \omega t$.
6. Writing the differential equation in the form $x'' + \omega_0^2 x = (F_0/m) \cos \omega_0 t$ with $\omega_0 = \sqrt{k/m}$, we see that it is the same as equation (13) in the text. Therefore a particular solution is $x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$. The general solution is $x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$. Imposing the initial conditions $x(0) = 0$ and $x'(0) = v_0$, we obtain $c_1 = 0$ and $\omega_0 c_2 = v_0$. Thus $c_2 = v_0/\omega_0$, and the solution is $x(t) = \left(\frac{v_0}{\omega_0} + \frac{F_0}{2m\omega_0} t\right) \sin \omega_0 t$.
7. $x_{\text{trial}} = A \cos 3t + B \sin 3t$ yields $(-9A + 12B + 4A) \cos 3t + (-9B - 12A + 4B) \sin 3t = 10 \cos 3t$. Thus $-5A + 12B = 10$ and $-5B - 12A = 0$. Thus $A = -50/169$ and $B = 120/169$. The steady periodic solution is $x_{sp}(t) = (-50/169) \cos 3t + (120/169) \sin 3t$. This equals $C \cos(3t - \alpha)$, where the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{50^2 + 120^2}/169 = 10/13$, and the phase shift is $\alpha = \pi - \tan^{-1}(12/5)$.
8. $x_{\text{trial}} = A \cos 5t + B \sin 5t$ yields $(-25A + 15B + 5A) \cos 5t + (-25B - 15A + 5B) \sin 5t = -4 \cos 5t$. Thus $-20A + 15B = -4$, and $-20B -$

$15A = 0$. We get $A = 16/125$ and $B = -12/125$. Thus the steady periodic solution is $x_{sp}(t) = (16/125)\cos 5t - (12/125)\sin 5t$. This equals $C\cos(5t-\alpha)$, where the amplitude is $C = \sqrt{A^2+B^2} = 4/25$, and $\alpha = 2\pi - \tan^{-1}(3/4)$.

15. $x_{\text{trial}} = A\cos\omega t + B\sin\omega t$ yields $(-\omega^2 A + 2\omega B + 2A)\cos\omega t + (-\omega^2 B - 2\omega A + 2B)\sin\omega t = 2\cos\omega t$. Thus $(2-\omega^2)A + 2\omega B = 2$ and $-2\omega A + (2-\omega^2)B = 0$. We find $A = \frac{2(2-\omega^2)}{4+\omega^4}$ and $B = \frac{4\omega}{4+\omega^4}$. The amplitude is $C(\omega) = \sqrt{A^2+B^2} = \frac{2}{\sqrt{4+\omega^4}}$. The critical points of this function are the critical points of $4+\omega^4$, namely solutions of $4\omega^3 = 0$, meaning the only crtical point is $\omega = 0$, where $C(0) = 1$. Thus $C(\omega)$ is a decreasing function for all $\omega$, and there is no practical resonance.