MATH 285 HOMEWORK 6 SOLUTIONS

Section 3.3

8. The characteristic equation is $r^2 - 6r + 13 = 0$. The roots are $r = \frac{6\pm\sqrt{-16}}{2} = 3 \pm 2i$, which are complex. The general solution is $y(x) = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x$.
14. The characteristic equation is $r^4 + 3r^2 - 4 = 0$. This polynomial factors as $(r^2 - 1)(r^2 + 4) = 0$, so the roots are $r = 1, -1, 2i, -2i$. The real root $r = 1$ gives a solution $y_1(x) = e^x$. The real root $r = -1$ gives a solution $y_2(x) = e^{-x}$, and the pair of complex solutions $r = \pm 2i$ gives a pair of solutions $y_3(x) = \cos 2x$ and $y_4(x) = \sin 2x$. The general solution is the linear combination of these four functions: $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$.
22. The characteristic equation is $9r^2 + 6r + 4 = 0$. The roots are $r = \frac{-6\pm\sqrt{-108}}{18} = \frac{-6\pm 6\sqrt{3}i}{18} = -(1/3) \pm i(1/\sqrt{3})$. The general solution is

$$y(x) = c_1 e^{-x/3} \cos(x/\sqrt{3}) + c_2 e^{-x/3} \sin(x/\sqrt{3}).$$

The derivative of this is

$$y(x) = c_1[(-1/3)e^{-x/3} \cos(x/\sqrt{3}) - e^{-x/3}(1/\sqrt{3}) \sin(x/\sqrt{3})]$$
$$+c_2[(-1/3)e^{-x/3} \sin(x/\sqrt{3}) + e^{-x/3}(1/\sqrt{3}) \cos(x/\sqrt{3})].$$

The initial condition $y(0) = 3$ yields $c_1 = 3$, and the condition $y'(0) = 4$ yields $-c_1/3 + c_2/\sqrt{3} = 4$. Thus $c_2 = 5\sqrt{3}$. The desired particular solution is

$$y(x) = 3e^{-x/3} \cos(x/\sqrt{3}) + 5\sqrt{3}e^{-x/3} \sin(x/\sqrt{3})$$

Section 3.4

13. (a) The characteristic equation is $10r^2+9r+2 = (5r+2)(2r+1) = 0$, and the roots are $r = -2/5, -1/2$. This are real and distinct, so the general solution is $x(t) = c_1 e^{-2t/5} + c_2 e^{-t/2}$. The initial conditions $x(0) = 0$, $x'(0) = 5$ yield the equations $c_1 + c_2 = 0$, $(-2/5)c_2 + (-1/2)c_2 = 5$. The solutions are $c_1 = 50, c_2 = -50$. Thus the particular solution is $x(t) = 50(e^{-2t/5} - e^{-t/2})$.
    (b) We are trying to find the maximum value of $x(t)$. The derivative is $x'(t) = -20e^{-2t/5} + 25e^{-t/2}$. Setting this equal to zero, we get $5e^{-t/10} = 4$. Thus $x'(t) = 0$ when $t = 10 \ln(5/4)$. Hence the mass's farthest distance to the right is $x(10 \ln(5/4)) = 512/125$.

24. In the critically damped case, we have $c^2 = 4km$. With $p = c/(2m)$, we may write the general solution as $x(t) = e^{-pt}(c_1 + c_2 t)$. The derivative is $x'(t) = (-p)e^{-pt}(c_1 + c_2 t) + e^{-pt}(c_2)$. Imposing the initial conditions $x(0) = x_0$ and $x'(0) = v_0$ yields the conditions $x_0 = c_1$, and $v_0 = -pc_1 + c_2$. Thus $c_1 = x_0$, and $c_2 = v_0 + px_0$. So the solution is $x(t) = e^{-pt}(x_0 + (v_0 + px_0)t) = (x_0 + v_0 t + px_0 t)e^{-pt}$.
27. In the overdamped case, $c^2 > 4km$, and if we write $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$, and $\gamma = (r_1 - r_2)/2$, then the general solution is $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. The velocity is $x'(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$. Imposing the initial conditions $x(0) = x_0$ and $x'(0) = v_0$ yields the equations $c_1 + c_2 = x_0$, and $c_1 r_1 + c_2 r_2 = v_0$. Multiply the first by r_1 and subtract it from the second to obtain $c_2 r_2 - c_2 r_1 = v_0 - r_1 x_0$. Thus $c_2 = (v_0 - r_1 x_0)/(r_2 - r_1) = (v_0 - r_1 x_0)/(-2\gamma)$. Plugging this back into the first equation we get $c_1 = (v_0 - r_2 x_0)/(2\gamma)$. Thus the solution is $x(t) = (1/2\gamma)[(v_0 - r_2 x_0)e^{r_1 t} - (v_0 - r_1 x_0)e^{r_2 t}]$.
30. In the underdamped case, we have $c^2 < 4km$. With $p = c/(2m)$, and $\omega_1 = \sqrt{(k/m)^2 - p^2}$, the general solution is $x(t) = e^{-pt}(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t)$. The derivative is $x'(t) = \omega_1 e^{-pt}(-c_1 \sin\omega_1 t + c_2 \cos\omega_1 t) - pe^{-pt}(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t)$. Imposing the initial conditions $x(0) = x_0$ and $x'(0) = v_0$, we get the equations $x_0 = c_1$, and $v_0 = \omega_1 c_2 - pc_1$. Thus $c_1 = x_0$, and $c_2 = (v_0 - px_0)/\omega_1$. Thus the solution is $x(t) = e^{-pt}(x_0 \cos\omega_1 t + ((v_0 + px_0)/\omega_1)\sin\omega_1 t)$.

Section 3.5

4. Try $y = Ae^x + Bxe^x$. We have $(4D^2 + 4D + 1)[e^x] = 9e^x$, and $(4D^2 + 4D + 1)[xe^x] = 9xe^x + 12e^x$. Thus $(4D^2 + 4D + 1)[Ae^x + Bxe^x] = A(9e^x) + B(9xe^x + 12e^x) = 9Bxe^x + (9A + 12B)e^x$. In order for this to equal $3xe^x$, we must have $9B = 3$ and $9A + 12B = 0$. Thus $B = 1/3$, and $A = -4/9$. The particular solution is $y_p(x) = (-4/9)e^x + (1/3)xe^x$.
6. Try $y = A + Bx + Cx^2$. We have $(2D^2 + 4D + 7)[1] = 7$, $(2D^2 + 4D + 7)[x] = 4 + 7x$, $(2D^2 + 4D + 7)[x^2] = 4 + 8x + 7x^2$. Thus $(2D^2 + 4D + 7)[A + Bx + Cx^2] = A(7) + B(4 + 7x) + C(4 + 8x + 7x^2) = 7Cx^2 + (8C + 7B)x + (4C + 4B + 7A)$. In order for this to equal x^2, we must have $7C = 1$, $7B + 8C = 0$, and $7A + 4B + 4C = 0$. Thus $C = 1/7$, $B = -8/49$, and $A = 4/343$. The particular solution is $y_p(x) = (4/343) - (8/49)x + (1/7)x^2$.
31. The complementary solution (general solution of homogeneous equation) is $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, because the roots of the characteristic equation $r^2 + 4 = 0$ are $r = \pm 2i$. To find a particular solution of the nonhomogeneous equation, we try $y = A + Bx$. We have $(D^2 + 4)[A + Bx] = 4A + 4Bx$. In order for this to equal $2x$, we take $A = 0$ and $B = 1/2$, so $y_p(x) = x/2$. The general solution of the nonhomogeneous equation is $y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + x/2$,

and its derivative is $y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x + 1/2$. Imposing the initial conditions $y(0) = 1$ and $y'(0) = 2$ yields $c_1 = 1$, and $2c_2 + 1/2 = 2$, whence $c_2 = 3/4$. The solution to the initial value problem is therefore $y(x) = \cos 2x + (3/4) \sin 2x + x/2$.

33. The complmentary solution is $y_c(x) = c_1 \cos 3x + c_2 \sin 3x$, because the roots of the characteristic equation $r^2 + 9 = 0$ are $r = \pm 3i$. To find a particular solution, try $y = A \cos 2x + B \sin 2x$. We have $(D^2 + 9)[A \cos 2x + B \sin 2x] = -4A \cos 2x - 4B \sin 2x + 9A \cos 2x + 9B \sin 2x = 5A \cos 2x + 5B \sin 2x$. In order for this to equal $\sin 2x$, we must have $A = 0$ and $5B = 1$. Thus $B = 1/5$ and $y_p(x) = (1/5) \sin 2x$. The general solution of the nonhomogeneous equation is $y(x) = c_1 \cos 3x + c_2 \sin 3x + (1/5) \sin 2x$, and its derivative is $y'(x) = -3c_1 \sin 3x + 3c_2 \cos 3x + (2/5) \cos 2x$. Imposing the initial conditions $y(0) = 1$ and $y'(0) = 0$ yields $1 = c_1$ and $0 = 3c_2 + 2/5$, whence $c_2 = -2/15$. The solution to the initial value problem is therefore $y(x) = \cos 3x - (2/15) \sin 3x + (1/5) \sin 2x$.