MATH 285 HOMEWORK 3 SOLUTIONS

Section 2.2

3. The critical points are the solutions of $f(x) = x^2 - 4x = 0$, which are $x = 0$ and $x = 4$. $f(x)$ is positive for $x < 0$ and $x > 4$, and $f(x)$ is negative for $0 < x < 4$. Thus $x = 0$ is a stable equilibrium, and $x = 4$ is an unstable equilibrium. Separating variables in equation gives

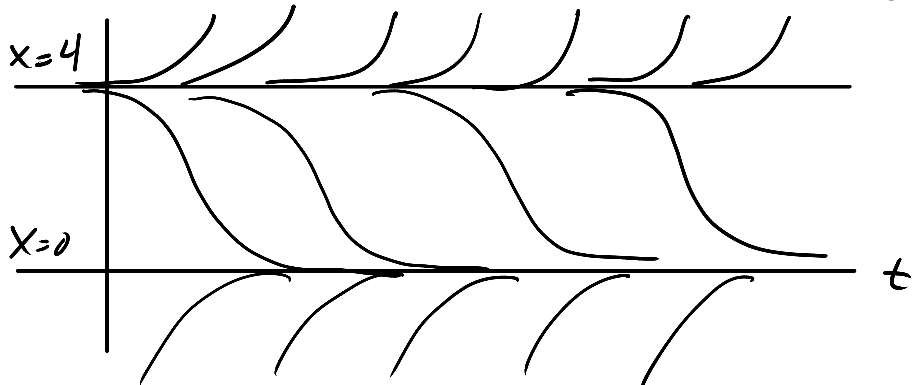
$$\int \frac{1}{x^2 - 4x} dx = \int dt$$

$$\frac{1}{4} \int \left(\frac{1}{x-4} - \frac{1}{x} \right) dx = \int dt$$

Thus $\ln|x-4| - \ln|x| = 4t + C$, or $\frac{x-4}{x} = Ce^{4t}$, where C is a nonzero constant. Writing the initial value as $x(0) = x_0$, we get $\frac{x_0 - 4}{x_0} = C$, leading to $\frac{x-4}{x} = \frac{x_0-4}{x_0} e^{4t}$. Solving for x gives

$$x(t) = \frac{4x_0}{x_0 + (4 - x_0)e^{4t}}$$

[Note: since this equation is precisely a logistic differential equation, with $M = 4$ and $k = -1$, one can also use the formula for the solution from section 2.1 rather than redoing the separation of variables.]



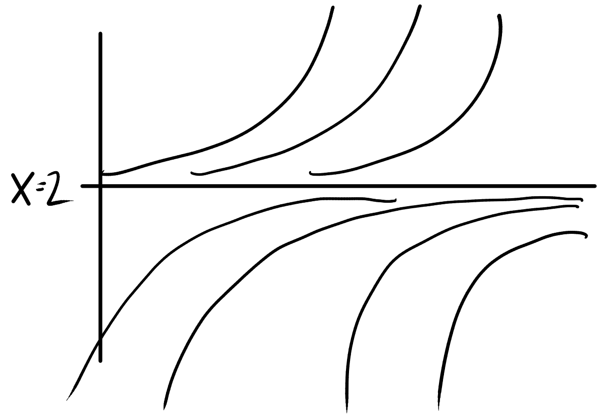
7. The critical point is the solution of $f(x) = (x-2)^2$, which is $x = 2$. Since $f(x)$ is positive for both $x < 2$ and $x > 2$, this equilibrium does not fit neatly into either the stable or unstable category. [Such equilibria are sometimes called semistable.] Separation of variables gives

$$\int \frac{1}{(x-2)^2} dx = \int dt$$

$$-(x-2)^{-1} = t + C$$

The initial value $x(0) = x_0$ gives $-(x_0-2)^{-1} = C$, leading to $-(x-2)^{-1} = t - (x_0-2)^{-1}$. Solving for x yields

$$x(t) = 2 + \frac{x_0 - 2}{1 - t(x_0 - 2)}$$



9. To find the critical points, we solve $f(x) = x^2 - 5x + 4 = 0$. Factoring gives $(x-4)(x-1) = 0$, so $x = 1$ and $x = 4$ are the critical points. $f(x)$ is positive for $x > 4$ and $x < 1$, and $f(x)$ is negative for $1 < x < 4$. Thus $x = 1$ is a stable equilibrium and $x = 4$ is an unstable equilibrium. Separation of variables gives

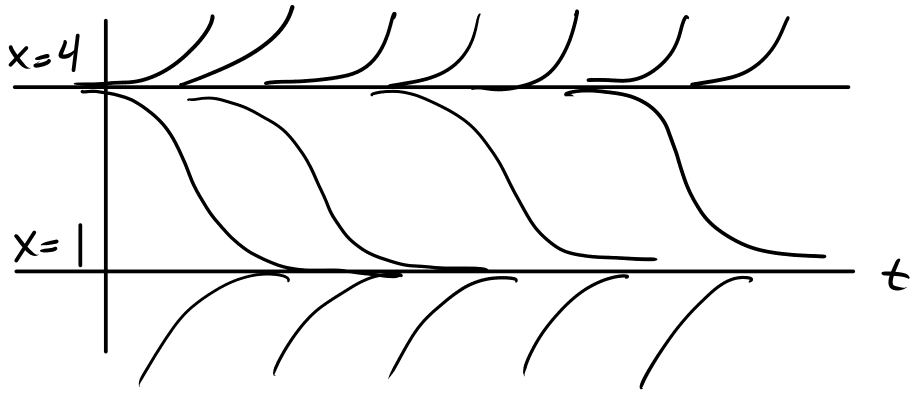
$$\int \frac{1}{(x-4)(x-1)} dx = \int dt$$

$$\frac{1}{3} \int \left(\frac{1}{x-4} - \frac{1}{x-1} \right) dx = \int dt$$

$$\ln|x-4| - \ln|x-1| = 3t + C$$

This leads to $\frac{x-4}{x-1} = Ce^{3t}$, where C is a nonzero constant. The initial condition $x(0) = x_0$ implies $\frac{x_0-4}{x_0-1} = C$, so $\frac{x-4}{x-1} = \frac{x_0-4}{x_0-1} e^{3t}$. Solving for x gives

$$x(t) = \frac{4(x_0-1) - (x_0-4)e^{3t}}{(x_0-1) - (x_0-4)e^{3t}}$$



28. We wish to verify that $kx(x-M)-h = k(x-H)(x-K)$, where $H = \frac{1}{2}(M + \sqrt{M^2+4h/k})$ and $K = \frac{1}{2}(M - \sqrt{M^2+4h/k})$. Expanding $k(x-H)(x-K)$ gives

$$k(x-H)(x-K) = kx^2 - k(H+K)x + kHK$$

From the formulas for H and K, we see $H+K = M$, and $HK = \frac{1}{4}(M^2 - \sqrt{M^2+4h/k}^2) = h/k$. Thus

$$k(x-H)(x-K) = kx^2 - k(H+K)x + kHK = kx^2 - kMx + h$$

On the other hand, if we expand $kx(x-M)+h$, we get $kx^2 - kMx + h$ as well.

Appendix

2.

$$\begin{aligned} y_0(x) &= 4 \\ y_1(x) &= 4 - 8x \\ y_2(x) &= 4 - 8x + 8x^2 \\ y_3(x) &= 4 - 8x + 8x^2 - (16/3)x^3 \\ y_4(x) &= 4 - 8x + 8x^2 - (16/3)x^3 + (8/3)x^4 \\ y(x) &= 4 - 8x + 8x^2 - (16/3)x^3 + (8/3)x^4 + \cdots = 4e^{-2x} \end{aligned}$$

4.

$$\begin{aligned} y_0(x) &= 2 \\ y_1(x) &= 2 + 2x^3 \\ y_2(x) &= 2 + 2x^3 + x^6 \\ y_3(x) &= 2 + 2x^3 + x^6 + (1/3)x^9 \\ y_4(x) &= 2 + 2x^3 + x^6 + (1/3)x^9 + (1/12)x^{12} \\ y(x) &= 2 + 2x^3 + x^6 + (1/3)x^9 + (1/12)x^{12} + \cdots = 2e^{(x^3)} \end{aligned}$$

10.

$$\begin{aligned} y_0(x) &= 0 \\ y_1(x) &= e^x - 1 \\ y_2(x) &= 2e^x - x - 2 \\ y_3(x) &= 3e^x - (1/2)x^2 - 2x - 3 \\ y(x) &= xe^x \end{aligned}$$

Section 2.4

2. Iterative formula: $y_{n+1} = y_n + h(2y_n)$. Approximate values 1.125 and 1.244. True value $y(1/2) \approx 1.359$.
6. Iterative formula: $y_{n+1} = y_n + h(-2x_n y_n)$. Approximate values 1.750 and 1.627. True value $y(1/2) \approx 1.558$.
10. Iterative formula: $y_{n+1} = y_n + h(2x_n y_n^2)$. Approximate values 1.125 and 1.231. True value $y(1/2) \approx 1.333$.