MATH 285 HOMEWORK 2 SOLUTIONS

Section 1.4

4. Separating variables gives $\int \frac{dy}{y} = \int \frac{4}{1+x} dx$. So $\ln|y| = 4\ln|1+x| + C$ or $y = \pm e^C |1+x|^4$. This can also be written $y = C(1+x)^4$, where C is an arbitrary nonzero constant. (One can also check that $C = 0$ gives a solution as well. So one could write the general solution as $y = C(1+x)^4$ with C any number. For the purposes of this problem, both forms are correct.)
12. Separating variables gives $\int \frac{y}{y^2+1} dy = \int x\, dx$. So $\frac{1}{2}\ln(y^2+1) = \frac{1}{2}x^2 + C$. Thus $y^2 + 1 = Ce^{x^2}$, or $y = \pm\sqrt{Ce^{x^2} - 1}$, where C is a constant.
20. Separating variables gives $\int \frac{1}{1+y^2} dy = \int 3x^2\, dx$. So $\tan^{-1} y = x^3 + C$. Using the initial condition $y(0) = 1$, we find that $\tan^{-1}(1) = 0 + C$, so $C = \pi/4$. The particular solution is $y = \tan(x^3 + \pi/4)$.

Section 1.5

9. We first rewrite the differential equation for $x > 0$ as $y' - \frac{1}{x}y = 1$. An integrating factor is $\rho = \exp(\int -\frac{1}{x} dx) = x^{-1}$, and multiplying by ρ gives $x^{-1}y' - x^{-2}y = x^{-1}$, or $\frac{d}{dx}(x^{-1}y) = x^{-1}$. Integrating leads to $x^{-1}y = \ln x + C$, so the general solution is $y = x\ln x + Cx$. The initial condition $y(1) = 7$ implies $C = 7$, so the particular solution is $y = x\ln x + 7x$.
13. An integrating factor is $\rho = \exp(\int 1\, dx) = e^x$. Multiplying by ρ gives $e^x y' + e^x y = e^{2x}$, or $\frac{d}{dx}(e^x y) = e^{2x}$. Integrating leads to $e^x y = \frac{1}{2}e^{2x} + C$, and $y = \frac{1}{2}e^x + Ce^{-x}$ is the general solution. The initial condition $y(0) = 1$ implies $C = 1/2$, so the particular solution is $y = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$. (This function is also known as $y = \cosh x$, the hyperbolic cosine.)
15. An integrating factor is given by $\rho = \exp(\int 2x\, dx) = e^{x^2}$. Multiplying by ρ gives $e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$, or $\frac{d}{dx}(e^{x^2}y) = xe^{x^2}$. Integrating leads to $e^{x^2}y = \frac{1}{2}e^{x^2} + C$, and the general solution is $y = \frac{1}{2} + Ce^{-x^2}$. The initial condition $y(0) = -2$ implies $C = -5/2$, So the particular solution is $y = \frac{1}{2} - \frac{5}{2}e^{-x^2}$.
33. Let $x(t)$ denote the amount of salt in kilograms in the tank after t seconds. In a small unit of time Δt, the change in the salt level, Δx, comes from the difference between the salt content of the water that is pumped in, which is zero, and the salt content of the water that

is pumped out, which is $(5/1000)x\Delta t$. Thus $\Delta x \approx -(5/1000)x\Delta t$, or $\frac{\Delta x}{\Delta t} = -(5/1000)x$. The corresponding differential equation is $\frac{dx}{dt} = -x/200$.

This equation is first-order linear; we write it as $\frac{dx}{dt}+(1/200)x = 0$. An integrating factor is $\rho = \exp(\int(1/200)dt) = e^{t/200}$. Multiplying we get $\frac{d}{dt}(e^{t/200}x) = 0$, so $e^{t/200}x = C$, and the general solution is $x(t) = Ce^{-t/200}$. (The equation is also separable, so one can use separation of variables to get this same answer.) The initial condition $x(0) = 100$ implies $C = 100$, so $x(t) = 100e^{-t/200}$.

The question actually asks to find the time when $x(t) = 10$. So we solve $10 = 100e^{-t/200}$, which gives $t = 200\ln 10$, which is approximately 461 seconds.

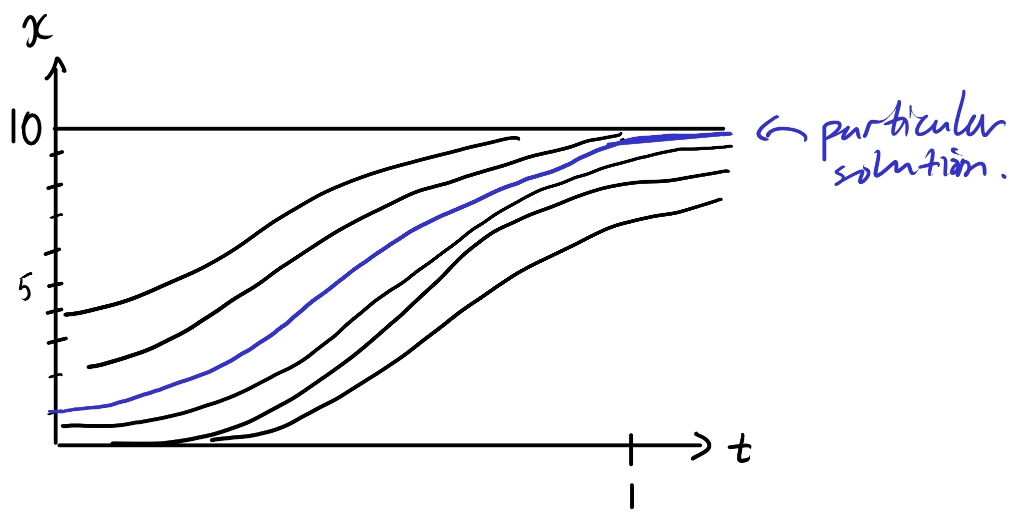
Section 2.1

2. Separating variables gives $\int \frac{1}{x(10-x)}dx = \int dt$. By the method of partial fractions

$$\int \frac{1}{x(10-x)}dx = \frac{1}{10}\int\left(\frac{1}{x} - \frac{1}{x-10}\right)dx = \frac{1}{10}(\ln|x| - \ln|x-10|)$$

So the general solution is $\ln|x|-\ln|x-10| = 10t+C$, or $\frac{x}{x-10} = Ce^{10t}$. The initial condition $x(0) = 1$ implies $C = -1/9$, leading to the particular solution $\frac{x}{10-x} = \frac{1}{9}e^{10t}$. Solving for x yields

$$x(t) = \frac{10e^{10t}}{e^{10t}+9} = \frac{10}{1+9e^{-10t}}$$



15. We can rewrite the differential equation as

$$\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right)$$

This shows that the limiting population is $M = \frac{a}{b}$. Using $D_0 = bP_0^2$ and $B_0 = aP_0$, we find

$$\frac{B_0 P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M$$

17. In terms of Problem 15, we have $k = b = D_0/P_0^2 = 12/(240)^2 = 1/4800$, and limiting population $M = B_0P_0/D_0 = 9 \cdot 240/12 = 180$, and $kM = 3/80$. From the solution of the logistic equation, we have

$$P(t) = \frac{180 \cdot 240}{240 + (180 - 240)e^{-3t/80}} = \frac{43200}{240 - 60e^{-3t/80}}$$

As 105% of M is $1.05 \cdot 180 = 189$, we must solve for t in

$$189 = \frac{43200}{240 - 60e^{-3t/80}}$$

This reduces to $e^{-3t/80} = 4/21$ and $t \approx 44.22$ months.