MATH 285 HOMEWORK 13 SOLUTIONS

SECTION 9.7

1. As usual, considering a separable solution $u(x, y) = X(x)Y(y)$ in Laplace's equation leads to equations of the form

$$\frac{d^2X}{dx^2} - \lambda X = 0, \quad \frac{d^2Y}{dy^2} + \lambda Y = 0$$

where λ is the separation constant. [Your answer may have $-\lambda$ in place of λ, that is, you may have written these equations with the sign of the λ-terms flipped. That is perfectly OK, but it will affect some of the later steps where we make assumptions about the sign of λ. -JP]

The boundary conditions $u(x, 0) = 0$ and $u(x, b) = 0$ translate into the endpoint conditions $Y(0) = 0$, $Y(b) = 0$. Thus we have an eigenvalue problem for Y:

$$\frac{d^2Y}{dy^2} + \lambda Y = 0, \quad Y(0) = 0, \quad Y(b) = 0$$

The eigenvalues and eigenfunctions are therefore

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad Y_n(y) = \sin\frac{n\pi y}{b}$$

The corresponding function $X_n(x)$ must satisfy

$$\frac{d^2X_n}{dx^2} - \left(\frac{n\pi}{b}\right)^2 X_n = 0$$

The general solution of this is

$$X_n(x) = A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}$$

[There is another way to write this general solution, namely as

$$X_n(x) = A_n \exp\left(\frac{n\pi x}{b}\right) + B_n \exp\left(-\frac{n\pi x}{b}\right)$$

It is perfectly fine to use this instead, although it will make some of the later steps look a little different. -JP] The boundary condition $u(0, y) = 0$ translates into $X_n(0) = 0$, so $A_n = 0$. Thus the n-th separable solution is

$$X_n(x)Y_n(y) = B_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b}$$

The general solution is the series

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

In order to satisfy the last boundary condition $u(a,y) = g(y)$, we must match

$$u(a,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = g(y)$$

This means that the quantity $B_n \sinh \frac{n\pi a}{b}$ must equal the n-th coefficient of the Fourier sine series of $g(y)$ on the interval $[0,b]$. That is, if we write

$$g(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b}, \quad b_n = \frac{2}{b} \int_0^b g(y) \sin \frac{n\pi y}{b}\, dy$$

we must have $B_n \sinh \frac{n\pi a}{b} = b_n$. Thus

$$B_n = \frac{b_n}{\sinh(n\pi a/b)}$$

The full solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)} \sin \frac{n\pi y}{b}$$

where b_n is the sine coefficient of $g(y)$ as above.

Section 10.1

3. The problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad hy(L) + y'(L) = 0$$

with domain $0 < x < L$ and $h > 0$ satisfies all parts of Theorem 1 on p. 638. Thus there are no negative eigenvalues.

To check $\lambda = 0$: $y(x) = Ax + B$. Endpoint condition $y'(0) = 0$ forces $A = 0$, so $y(x) = B$ is a constant function. But then $hy(L) + y'(L) = hB + 0 = 0$ forces $B = 0$ as well, so $\lambda = 0$ is not an eigenvalue.

To check $\lambda > 0$: write $\lambda = \alpha^2$. Then $y(x) = A\cos\alpha x + B\sin\alpha x$, and $y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x$. The condition $y'(0) = 0$ means $B\alpha = 0$, forcing $B = 0$, and so $y(x) = A\cos\alpha x$. The condition $hy(L) + y'(L) = 0$ becomes

$$hA\cos\alpha L - A\alpha\sin\alpha L = 0$$

This forces $A = 0$ as well unless $h\cos\alpha L - \alpha\sin\alpha L = 0$. Setting $\beta = \alpha L$, we can write this condition as

$$h\cos\beta = (\beta/L)\sin\beta,$$

$$\beta\tan\beta = hL,$$

or

$$\tan\beta = hL/\beta$$

One can see from a graph that there are infinitely many positive solutions of this equation $\beta_1 < \beta_2 < \beta_3 < \cdots$. The eigenvalues are therefore

$$\lambda_n = \alpha_n^2 = (\beta_n/L)^2$$

and an eigenfunction for λ_n is

$$y_n(x) = \cos\alpha_n x = \cos\frac{\beta_n x}{L}$$

To estimate the values of β_n for large n, note that if β is large, then hL/β is close to zero. Therefore, the solution of $\tan\beta = hL/\beta$ is close to a solution of $\tan\beta = 0$, which would be $n\pi$ for an integer n. In fact, if one is careful about the indexing one finds $\beta_n \approx (n-1)\pi$ for large n.

8. This is a continuation of the previous problem. By equations (23), (25) on p. 641 (with $r(x) = 1$), there is an eigenfunction series

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sum_{n=1}^{\infty} c_n \cos\frac{\beta_n x}{L}$$

where the coefficients are given by

$$c_n = \frac{\int_0^L f(x) y_n(x)\,dx}{\int_0^L y_n(x)^2\,dx} = \frac{\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx}{\int_0^L \cos^2\frac{\beta_n x}{L}\,dx}$$

We compute for $f(x) = 1$:

$$\int_0^L 1\cdot\cos\frac{\beta_n x}{L}\,dx = \left[\frac{L}{\beta_n}\sin\frac{\beta_n x}{L}\right]_0^L = \frac{L}{\beta_n}\sin\beta_n$$

$$\int_0^L \cos^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}(1+\cos\frac{2\beta_n x}{L})\,dx$$

$$= \frac{1}{2}\left[x + \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right]_0^L = \frac{1}{2}\left(L + \frac{L}{2\beta_n}\sin 2\beta_n\right)$$

So

$$c_n = \frac{\frac{L}{\beta_n}\sin\beta_n}{\frac{1}{2}\left(L + \frac{L}{2\beta_n}\sin 2\beta_n\right)} = \frac{4\sin\beta_n}{2\beta_n + \sin 2\beta_n}$$

The desired eigenfunction expansion is

$$1 = \sum_{n=1}^{\infty} \frac{4\sin\beta_n}{2\beta_n + \sin 2\beta_n}\cos\frac{\beta_n x}{L} \quad (0 < x < L)$$