MATH 285 HOMEWORK 12 SOLUTIONS

Section 9.6

1. The general solution of the problem $y_{tt} = 4y_{xx}$, $y(0,t) = 0$, $y(\pi, t) = 0$ is

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \cos 2nt + B_n \sin 2nt) \sin nx$$

This is obtained from the general solution derived in the lecture by setting $c = 2$ and $L = \pi$. We have to match the initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \frac{1}{10} \sin 2x$$

$$y_t(x,0) = \sum_{n=1}^{\infty} 2nB_n \sin nx = 0$$

Thus we take $A_2 = 1/10$, all other $A_n = 0$, and all $B_n = 0$. The final result is

$$y(x,t) = \frac{1}{10} \cos 4t \sin 2x$$

3. In this problem we use the same general solution with $c = \sqrt{1/4} = 1/2$ and $L = \pi$. Thus

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{nt}{2} + B_n \sin \frac{nt}{2}) \sin nx$$

We have to match the initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \frac{1}{10} \sin x$$

$$y_t(x,0) = \sum_{n=1}^{\infty} \frac{n}{2} B_n \sin nx = \frac{1}{10} \sin x$$

Thus we take $A_1 = 1/10$, all other $A_n = 0$, and $B_1 = 1/5$, all other $B_n = 0$. The result is

$$y(x,t) = \left(\frac{1}{10} \cos \frac{t}{2} + \frac{1}{5} \sin \frac{t}{2} \right) \sin x$$

5. In this problem $c = 5$ and $L = 3$. Thus the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{5n\pi t}{3} + B_n \sin \frac{5n\pi t}{3} \right) \sin \frac{n\pi x}{3}$$

This needs to match the initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{3} = \frac{1}{4}\sin \pi x$$

$$y_t(x,0) = \sum_{n=1}^{\infty} \frac{5n\pi}{3} B_n \sin\frac{n\pi x}{3} = 10 \sin 2\pi x$$

In the first equation, we see that the term on the right corresponds to $n = 3$, so we take $A_3 = 1/4$ and all other $A_n = 0$. In the second equation, the term on the right corresponds to $n = 6$, so we take $B_n = 0$ for all n other than $n = 6$, whereas

$$\frac{5 \cdot 6\pi}{3} B_6 = 10$$

Thus $B_6 = 1/\pi$. The result is

$$y(x,t) = \frac{1}{4}\cos 5\pi t \sin \pi x + \frac{1}{\pi}\sin 10\pi t \sin 2\pi x$$

8. In this problem $c = 2$ and $L = \pi$, so the general solution is

$$y(x,t) = \sum_{n=1}^{\infty}(A_n \cos 2nt + B_n \sin 2nt)\sin nx$$

We have to match the initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin x$$

$$y_t(x,0) = \sum_{n=1}^{\infty} 2nB_n \sin nx = 1$$

The first equation means we take $A_1 = 1$ and all other $A_n = 0$. The second equation means that the number $2nB_n$ must be the nth coefficient of the sine series for the function 1 on the interval $0 < x < \pi$. The odd periodic extension is a square wave, and its Fourier coefficients are $b_n = 4/(n\pi)$ for odd n, and $b_n = 0$ for even n.

$$2nB_n = 4/(n\pi), \text{ for odd } n;$$
$$2nB_n = 0, \text{ for even } n.$$

Thus $B_n = 0$ for even n, and $B_n = 2/(n^2\pi)$ for odd n. The result is

$$y(x,t) = \cos 2t \sin x + \sum_{n \text{ odd}} \frac{2}{n^2\pi}\sin 2nt \sin nx$$

16. Suppose $u = x + at$ and $v = x - at$. Thus $x = (u+v)/2$ and $t = (u-v)/(2a)$. Thus we compute $\partial y/\partial u$

$$\frac{\partial y}{\partial u} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial u} = \frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}$$

$$\frac{\partial^2 y}{\partial v \partial u} = \frac{\partial}{\partial v}\left[\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right] = \frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right]\frac{\partial x}{\partial v} + \frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right]\frac{\partial t}{\partial v}$$
$$= \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right] - \frac{1}{2a}\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right]$$
$$= \frac{1}{4}\frac{\partial^2 y}{\partial x^2} + \frac{1}{4a}\frac{\partial^2 y}{\partial x \partial t} - \frac{1}{4a}\frac{\partial^2 y}{\partial t \partial x} - \frac{1}{4a^2}\frac{\partial^2 y}{\partial t^2}$$
$$= \frac{1}{4}\frac{\partial^2 y}{\partial x^2} - \frac{1}{4a^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{4a^2}\left[a^2\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2}\right]$$

Thus

$$a^2\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0 \text{ if and only if } \frac{\partial^2 y}{\partial v \partial u} = 0$$

That is to say, with respect to the coordinates (u, v), the wave equation is $y_{uv} = 0$.

Next, suppose that $y(u, v)$ is a solution of the wave equation, thus $y_{uv} = 0$, or

$$\frac{\partial}{\partial v}(y_u) = 0$$

This means that $y_u(u, v)$ is a function that does not depend on v, but which may depend on u. Thus $y_u = f(u)$ for some function $f(u)$. Now we have the equation

$$\frac{\partial y}{\partial u} = f(u)$$

By integrating, we find that

$$y(u, v) = \int f(u)\, du + G(v)$$

where the constant of integration with respect to u may depend on v. Letting $F(u) = \int f(u)\, du$, we have found that $y(u, v)$ must have the form

$$y(u, v) = F(u) + G(v)$$

Changing back to (x, t) by the formulas $u = x + at$, $v = x - at$, we find

$$y(x, t) = F(x + at) + G(x - at)$$

Thus, every solution of the wave equation $y_{tt} = a^2 y_{xx}$ has this form, the sum of a left-moving and a right-moving wave at speed a.