MATH 285 HOMEWORK 11 SOLUTIONS

Section 9.5

1. Since this is a heat equation problem with zero endpoint temperatures, we konw from Theorem 1 on page 604, with $L = \pi$ and $k = 3$, that
$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin nx$$
For some coefficients b_n. The initial condition $u(x,0) = 4\sin 2x$ becomes
$$4\sin 2x = \sum_{n=1}^{\infty} b_n \sin nx$$
This equation is satisfied if $b_2 = 4$ and $b_n = 0$ for all $n \neq 4$. Thus the solution is
$$u(x,t) = 4e^{-12t} \sin 2x$$

3. This is a heat equation problem with zero endpoint temperatures, so we know from Theorem 1, with $L = 1$ and $k = 2$, that
$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-2n^2\pi^2 t} \sin n\pi x$$
The initial condition $u(x,0) = 5\sin \pi x - \frac{1}{5}\sin 3\pi x$ becomes
$$5\sin \pi x - \frac{1}{5}\sin 3\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$
Thus we must take $b_1 = 5$, $b_3 = -1/5$, and all other $b_n = 0$. The solution thus contains only the $n = 1$ and $n = 3$ terms, and is
$$u(x,t) = 5e^{-2\pi^2 t} - \frac{1}{5}e^{-18\pi^2 t} \sin 3\pi x$$

5. This is a heat equation problem with insulated ends, so we know from Theorem 2 on page 607, with $L = 3$ and $k = 2$, that
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{\frac{-2n^2\pi^2 t}{9}} \cos \frac{n\pi x}{3}$$
The initial condition $u(x,0) = 4\cos\frac{2\pi x}{3} - 2\cos\frac{4\pi x}{3}$ becomes
$$4\cos\frac{2\pi x}{3} - 2\cos\frac{4\pi x}{3} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}$$

Thus we must take $a_0 = 0$, $a_2 = 4$, $a_4 = -2$, and all other $a_n = 0$. The solution only contains the $n = 2$ and $n = 4$ terms, and is

$$u(x,t) = 4e^{\frac{-8\pi^2 t}{9}} \cos\frac{2\pi x}{3} - 2e^{\frac{-32\pi^2 t}{9}} \cos\frac{4\pi x}{3}$$

10. This is a zero endpoint temperature problem with $L = 10$ and $k = 1/5$, so we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/500} \sin\frac{n\pi x}{10}$$

The initial condition is

$$4x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

We recognize this as the relation for the sine series of the function $4x$ defined on the interval $0 < x < 10$. Thus

$$b_n = \frac{2}{10}\int_0^{10} 4x \sin\frac{n\pi x}{10}\,dx = \frac{4}{5}\left[-\frac{10}{n\pi}x\cos\frac{n\pi x}{10} + \left(\frac{10}{n\pi}\right)^2 \sin\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{4}{5}\left[-\frac{100}{n\pi}\cos n\pi\right] = -\frac{80}{\pi}\frac{1}{n}\cos n\pi = \frac{80}{\pi}\frac{(-1)^{n+1}}{n}$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{80}{\pi}\frac{(-1)^{n+1}}{n} e^{-n^2\pi^2 t/500} \sin\frac{n\pi x}{10}$$

11. This is an insulated ends problem with $L = 10$ and $k = 1/5$, so we have

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/500} \cos\frac{n\pi x}{10}$$

The initial condition is

$$4x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{10}$$

We recognize this as the cosine series of the function $4x$ defined on the interval $0 < x < 10$. Thus

$$a_0 = \frac{2}{10}\int_0^{10} 4x\,dx = \frac{1}{5}[2x^2]_0^{10} = 40$$

$$a_n = \frac{2}{10}\int_0^{10} 4x \cos\frac{n\pi x}{10}\,dx = \frac{4}{5}\left[\frac{10}{n\pi}x\sin\frac{n\pi x}{10} + \left(\frac{10}{n\pi}\right)^2 \cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{4}{5}\left[\frac{100}{n^2\pi^2}(\cos n\pi - 1)\right] = \frac{80}{n^2\pi^2}((-1)^n - 1)$$

The quantity $(-1)^n - 1$ is 0 for even n and -2 for odd n, thus

$$a_n = -\frac{160}{n^2\pi^2} \text{ if } n \text{ is odd}, \quad a_n = 0 \text{ if } n \text{ is even}.$$

Using these values for a_n, the solution is

$$u(x,t) = 20 - \frac{160}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-n^2\pi^2 t/500} \cos\frac{n\pi x}{10}$$

17. Consider a rod with initial temperature $u(x,0) = f(x)$ and fixed endpoint temperatures $u(0,t) = A$ and $u(L,t) = B$.
    (a) The steady state solution $u_{\text{ss}}(x)$ does not depend on t, and it satisfies $\frac{\partial^2 u_{\text{ss}}}{\partial x^2} = 0$, $u_{\text{ss}}(0) = A$, and $u_{\text{ss}}(L) = B$. Integrating $\frac{\partial^2 u_{\text{ss}}}{\partial x^2} = 0$ twice with respect to x gives $u_{\text{ss}}(x) = cx + d$. The endpoint conditions then become

    $$A = c(0) + d, \quad B = cL + d$$

    The solution of which is $d = A$, $c = (B - A)/L$. Thus

    $$u_{\text{ss}}(x) = (B - A)\frac{x}{L} + A$$

    (b) The transient temperature is defined to be

    $$u_{\text{tr}}(x,t) = u(x,t) - u_{\text{ss}}(x).$$

    Given that u satisfies the equation

    $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

    and what we know about $u_{\text{ss}}(x)$, we find

    $$\frac{\partial u_{\text{tr}}}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial u_{\text{ss}}}{\partial t} = \frac{\partial u}{\partial t} - 0 = \frac{\partial u}{\partial t}$$

    $$\frac{\partial^2 u_{\text{tr}}}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_{\text{ss}}}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - 0 = \frac{\partial^2 u}{\partial x^2}$$

    We find that u_{tr} also satisfies the equation

    $$\frac{\partial u_{\text{tr}}}{\partial t} = k\frac{\partial^2 u_{\text{tr}}}{\partial x^2}$$

    On the other hand, since $u(0,t) = A$, and $u_{\text{ss}}(0,t) = A$, we find

    $$u_{\text{tr}}(0,t) = u(0,t) - u_{\text{ss}}(0,t) = A - A = 0.$$

    Similarly, since $u(L,t) = B$ and $u_{\text{ss}}(L,t) = B$, we have $u_{\text{tr}}(L,t) = 0$. Finally, since $u(x,0) = f(x)$, while $u_{\text{ss}}(x) = (B-A)(x/L)+A$ for all values of t, we have

    $$u_{\text{tr}}(x,0) = u(x,0) - u_{\text{ss}}(x) = f(x) - u_{\text{ss}}(x) = f(x) - [(B - A)(x/L) + A]$$

    Thus $u_{\text{tr}}(x,t)$ is in fact a solution of the boundary value problem

    $$\frac{\partial u_{\text{tr}}}{\partial t} = k\frac{\partial^2 u_{\text{tr}}}{\partial x^2}, \quad u_{\text{tr}}(0,t) = u_{\text{tr}}(L,t) = 0, \quad u_{\text{tr}}(x,0) = f(x) - u_{\text{ss}}(x).$$

(c) We can interpret the previous part as saying that $u_{\text{tr}}(x,t)$ is a solution of the zero endpoint temperature problem with initial temperature distribution $f(x) - u_{\text{ss}}(x)$. Therefore, Theorem 1 from page 604 applies to $u_{\text{tr}}(x,t)$, telling us that

$$u_{\text{tr}}(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$$

where c_n are the coefficients of the sine series of the initial temperature distribution $f(x) - u_{\text{ss}}(x)$, namely

$$c_n = \frac{2}{L} \int_0^L [f(x) - u_{\text{ss}}(x)] \sin \frac{n\pi x}{L}\, dx$$

Since $u(x,t) = u_{\text{tr}}(x,t) + u_{\text{ss}}(x)$, we get finally

$$u(x,t) = u_{\text{ss}}(x) + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$$

where c_n are given as above.

Note: Using the known form of $u_{\text{ss}}(x)$, we can write this even more explicitly as

$$u(x,t) = A + (B - A)\frac{x}{L} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}$$

with coefficients c_n given by

$$c_n = \frac{2}{L} \int_0^L [f(x) - (B - A)(x/L) - A] \sin \frac{n\pi x}{L}\, dx$$