MATH 285 HOMEWORK 10 SOLUTIONS

Section 9.4

4. Taking the result of Example 1 from Section 9.3 with $L = 2$, and then multiplying by 2, we find that the Fourier series of the even periodic function of period 4 such that $F(t) = 2t$ for $0 < t < 2$ is given by

$$F(t) = 2 - \frac{16}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{n\pi t}{2}$$

To solve $x'' + 4x = F(t)$, we use a trial solution that is a cosine series: $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2}$. This leads to

$$-\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} a_n \cos \frac{n\pi t}{2} + 2a_0 + 4\sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2} = 2 - \frac{16}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{n\pi t}{2}$$

Comparing coefficients, we find $2a_0 = 2$ and $a_n = 0$ for even n, while for odd n,

$$\left(-\frac{n^2\pi^2}{4} + 4 \right) a_n = -\frac{16}{\pi^2 n^2}.$$

Thus $a_0 = 1$, and for odd n,

$$a_n = \frac{-16/\pi^2 n^2}{4 - \pi^2 n^2/4} = -\frac{64}{\pi^2 n^2 (16 - \pi^2 n^2)}$$

Thus the Fourier series solution for the steady periodic solution $x_{\text{sp}}(t)$ is

$$x_{\text{sp}}(t) = \frac{1}{2} - 64 \sum_{n \text{ odd}} \frac{\cos n\pi t/2}{\pi^2 n^2 (16 - \pi^2 n^2)}$$

7. Since $m = 1$ and $k = 9$, the natural frequency is $\omega_0 = \sqrt{9} = 3$. The Fourier series for $F(t)$ (which is a squarewave) is

$$F(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin nt = \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \cdots \right)$$

Since the Fourier series contains a $\sin 3t$ term, resonance will occur in this system.

8. Since $m = 2$ and $k = 10$, the natural frequency is $\omega_0 = \sqrt{5}$. Now since $F(t)$ is an odd periodic function of period 2, its Fourier series is a sine series,

$$F(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

In this problem, we can see that resonance does not occur, even without knowing the coefficients b_n. In order for resonance to occur, there would need to be an integer n so that $n\pi = \omega_0 = \sqrt{5}$. That would mean that $\sqrt{5}/\pi$ would have to be an integer. But $\sqrt{5}/\pi \approx 0.711763$ is not an integer, therefore resonance does not occur.

Section 3.8

[*Note:* In the following problems we never consider the case $\lambda < 0$ because the statement of the problem asserts that all of the eigenvalues are nonnegative, and we take this as a given. Strictly speaking, when one is considering an eigenvalue problem from first principles, one needs to also check whether $\lambda < 0$ can be an eigenvalue. –JP]

1. The case $\lambda = 0$: $y'' = 0$ implies $y(x) = A + Bx$. The endpoint condition $y'(0) = 0$ implies $B = 0$, so $y(x) = A$. Then $y(1) = 0$ implies $A = 0$ as well. Thus the only solution is $y(x) = 0$, and $\lambda = 0$ is not an eigenvalue.

   The case $\lambda > 0$: For convenience let $\alpha = \sqrt{\lambda}$, so $\lambda = \alpha^2$. Then the equation $y'' + \alpha^2 y = 0$ implies $y(x) = A\cos\alpha x + B\sin\alpha x$. Thus $y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x$. The conditions $y'(0) = 0$ and $y(1) = 0$ then imply $B = 0$, and $A\cos\alpha + B\sin\alpha = 0$. This simplifies to $B = 0$ and $A\cos\alpha = 0$. Thus A is forced to be zero unless $\cos\alpha = 0$. This happens precisely when α is an odd multiple of $\pi/2$, $\alpha = (2n-1)\pi/2$, $n = 1, 2, 3, \ldots$, and in this case $y(x) = \cos\alpha x$ is a nonzero solution to the endpoint problem. Thus, the positive eigenvalues are $\lambda = \alpha^2 = (2n-1)^2\pi^2/4$, $n = 1, 2, 3, \ldots$, and the associated eigenfunctions are $y_n(x) = \cos[(2n-1)\pi x/2]$.
2. The case $\lambda = 0$: $y'' = 0$ implies $y(x) = A + Bx$. The endpoint condition $y'(0) = 0$ implies $B = 0$, so $y(x) = A$. Then other endpoint condition $y'(\pi) = 0$ is then also satisified. So $\lambda = 0$ is an eigenvalue, and the associated eigenfunction is the constant function $y_0(x) = 1$.

   The case $\lambda > 0$: Write $\lambda = \alpha^2$. The equation $y'' + \alpha^2 y = 0$ implies $y(x) = A\cos\alpha x + B\sin\alpha x$. The condition $y'(0) = 0$ implies $\alpha B = 0$, so $B = 0$, and $y(x) = A\cos\alpha x$. The other endpoint condition $y'(\pi) = 0$ implies that $-\alpha A\sin\alpha\pi = 0$. This equation forces $A = 0$ be zero unless $\sin\alpha\pi = 0$. This happens when $\alpha\pi = n\pi$ for some integer n, that is, when $\alpha = n$ is an integer, $n = 1, 2, 3, \ldots$. Thus the positive eigenvalues are $\lambda = \alpha^2 = n^2$ for $n = 1, 2, 3, \ldots$, and the associated eigenfunctions are $y_n(x) = \cos nx$.
4. The case $\lambda = 0$: $y'' = 0$ implies $y(x) = A + Bx$. The endpoint condition $y'(-\pi) = 0$ implies $B = 0$, and then the condition $y'(\pi) = 0$ is also satisfied. So $\lambda = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = 1$.

   The case $\lambda > 0$: Write $\lambda = \alpha^2$. The general solution is $y(x) = A\cos\alpha x + B\sin\alpha x$, with derivative $y'(x) = -\alpha A\sin\alpha x + \alpha B\cos\alpha x$.

Then condition $y'(-\pi) = 0$ implies

$$A \sin \alpha\pi + B \cos \alpha\pi = 0$$

The condition $y'(\pi) = 0$ implies

$$-A \sin \alpha\pi + B \cos \alpha\pi = 0$$

Taking the sum and difference of these two equations we get the equivalent set of conditions

$$A \sin \alpha\pi = 0 \text{ and } B \cos \alpha\pi = 0$$

If $A \neq 0$, then $\sin \alpha\pi = 0$, so $\alpha = n$ is an integer, and then $\cos n\pi \neq 0$, so $B = 0$. If $B \neq 0$, then $\cos \alpha\pi = 0$, so $\alpha\pi$ is an odd multiple of $\pi/2$ which is to say that α is an odd multiple of $1/2$, and the $\sin \alpha\pi \neq 0$, so $A = 0$.

There are thus two families of α-values, the integers and the odd multiples of $1/2$. This is the same as *all* integer multiples of $1/2$. Thus the positive eigenvalues are $\lambda = \alpha^2 = (n/2)^2$ for $n = 1, 2, 3, \ldots$. The associated eigenfunction is $y_n(x) = \cos(nx/2)$ if n is even and $y_n(x) = \sin(nx/2)$ if n is odd.

5. The case $\lambda = 0$: $y'' = 0$ implies $y(x) = A + Bx$. The condition $y'(2) = 0$ implies $B = 0$, so $y(x) = A$, and then the condition $y(-2) = 0$ implies $A = 0$ as well. So $\lambda = 0$ is not an eigenvalue.

   The case $\lambda > 0$: Write $\lambda = \alpha^2$. The general solution is $y(x) = A \cos \alpha x + B \sin \alpha x$. The condition $y(-2) = 0$ implies $A \cos 2\alpha - B \sin 2\alpha = 0$. The condition $y'(2) = 0$ implies $-\alpha A \sin 2\alpha + \alpha B \cos 2\alpha = 0$. This yields the system

$$A \cos 2\alpha = B \sin 2\alpha \text{ and } A \sin 2\alpha = B \cos 2\alpha$$

   Note that if $A = 0$, then $B = 0$ as well, and vice versa. So if $A \neq 0$ then $B \neq 0$, and vice versa. Now, taking the ratio of these two equations, we find $1/\tan 2\alpha = \tan 2\alpha$, or $(\tan 2\alpha)^2 = 1$. Thus there are two possibilies $\tan 2\alpha = 1$ or $\tan 2\alpha = -1$. The solutions of $\tan 2\alpha = 1$ occur when $2\alpha = \pi/4 + n\pi$ for some integer $n = 0, 1, 2, \ldots$, while the solutions of $\tan 2\alpha = -1$ occur when $2\alpha = 3\pi/4 + n\pi$ for $n = 0, 1, 2, \ldots$. Combining the cases, we see that 2α can be any odd multiple of $\pi/4$

$$2\alpha = (2n-1)\frac{\pi}{4} \quad n = 1, 2, 3, \ldots$$

   Thus $\alpha = (2n-1)\pi/8$, and $\lambda = (2n-1)^2\pi^2/64$, for $n = 1, 2, 3, \ldots$.

   For odd n, the associated eigenfunction comes from the case $\tan 2\alpha = 1$, which means $A = B$. Thus it is $y_n(x) = \cos \alpha_n x + \sin \alpha_n x$, where $\alpha_n = (2n-1)\pi/8$, and n is odd.

   For even n, the associated eigenfunction comes from the case $\tan 2\alpha = -1$, which means $A = -B$. Thus it is $y_n(x) = \cos \alpha_n x - \sin \alpha_n x$, where $\alpha_n = (2n-1)\pi/8$, and n is even.

6. The case $\lambda = 0$: $y'' = 0$ implies $y(x) = A + Bx$, and the condition $y'(0) = 0$ implies $B = 0$. while the conditon $y(1) + y'(1) = 0$ implies $A + B + B = 0$. Thus $A = B = 0$, and $\lambda = 0$ is not an eigenvalue.

   The case $\lambda > 0$: Write $\lambda = \alpha^2$. The general solution is $y(x) = A\cos\alpha x + B\sin\alpha x$. Then $y'(x) = -\alpha A\sin\alpha x + \alpha B\cos\alpha x$. The condition $y'(0) = 0$ implies $B = 0$, so $y(x) = A\cos\alpha x$, $y'(x) = -\alpha A\sin\alpha x$. The condition $y(1)+y'(1) = 0$ the implies that $A\cos\alpha - \alpha A\sin\alpha = 0$. This forces $A = 0$ unless α satisfies the condition $\cos\alpha - \alpha\sin\alpha = 0$. This equation can be rewritten as $\tan\alpha = 1/\alpha$. As the graph in the textbook illustrates, this equation does have infinitely many positive solutions for α. Since they are difficult to calculate we just denote them by α_n, $n = 1, 2, 3, \ldots$. The corresponding positive eigenvalues $\lambda_n = \alpha_n^2$, and the corresponding eigenfunctions are $y_n(x) = \cos\alpha_n x$.

   [*Note*: Entering the command `Solve[Tan[z] == 1/z, z]` into Mathematica yields the response "This system cannot be solved with the methods available to Solve." So you shouldn't feel bad that you can't solve this equation. –JP]