NAME: Solutions

NetID:

MATH 285 G1 Final Exam (A) **May 10, 2016** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
EC	20	
Total	200	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- **No books, notes, or calculators**. You are not permitted to use anything other than a writing utensil.
- You have three hours.

Orthogonality formulas

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{1}$$

$$\int_{-L}^{L} \sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2}$$

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = 0 \tag{3}$$

Some integral formulas

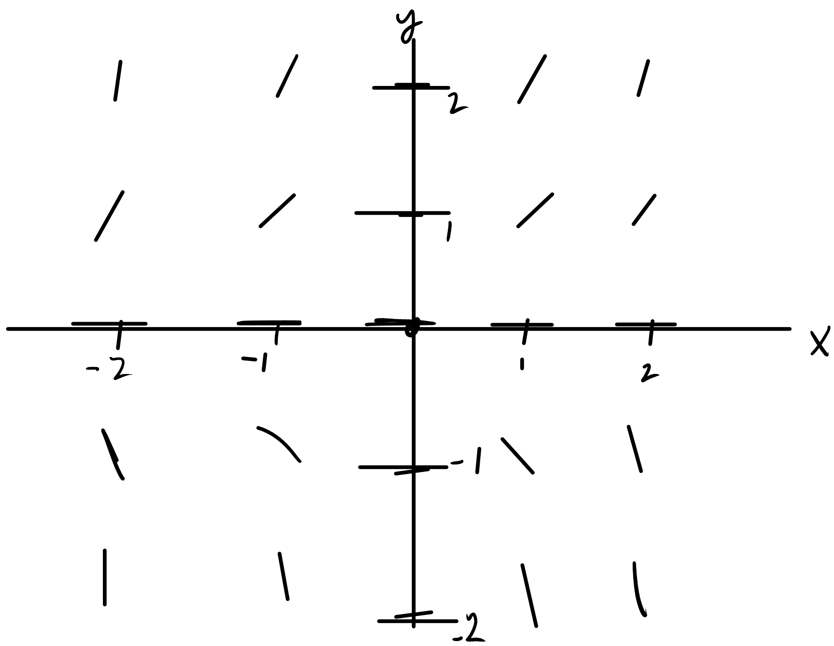
$$\int u\cos u\,du = u\sin u + \cos u + C \tag{4}$$

$$\int u\sin u\,du = -u\cos u + \sin u + C \tag{5}$$

1. (20 points) Draw a slope field for the differential equation

$$\frac{dy}{dx} = x^2 y.$$

Please draw about 16-20 slopes spread over **all four quadrants** of the xy-plane.



2. (20 points) A carbon ball has an initial temperature of $30°C$ at time $t = 0$. It is submerged in a vat of molten iron at $1500°C$. Because the melting point of carbon is around $3500°C$, it will not melt but rather heat up to the temperature of the iron, in accordance with Newton's law of cooling

$$\frac{dT}{dt} = -1.7(T - 1500),$$

where T is the temperature of the ball. Find the amount of time it takes for the ball to reach a temperature of $1000°C$. Your answer does not need to be simplified.

$$\int \frac{dT}{T-1500} = \int -1.7\,dt$$

$$\ln|T-1500| = -1.7t + C$$

$$T - 1500 = Ce^{-1.7t}$$

$$T = 1500 + Ce^{-1.7t}$$

$$T(0) = 30$$

$$30 = 1500 + Ce^{0}$$

$$30 - 1500 = C$$

$$C = -1470$$

$$T(t) = 1500 - 1470e^{-1.7t}$$

When is $T(t) = 1000$?

$$1000 = 1500 - 1470e^{-1.7t}$$

$$-500 = -1470e^{-1.7t}$$

$$\frac{50}{147} = e^{-1.7t}$$

$$\ln\left(\frac{50}{147}\right) = -1.7t$$

$$\boxed{t = \frac{1}{-1.7}\ln\left(\frac{50}{147}\right) \text{ or } \frac{1}{1.7}\ln\left(\frac{147}{50}\right)}$$

3. (20 points) Find the solution of the initial value problem

$$y' - 5y = 3e^{5x}, \quad y(0) = 0.$$

Integrating factor method:

$$\rho = e^{\int -5\,dx} = e^{-5x}$$

$$e^{-5x} y' - 5e^{-5x} y = 3$$

$$\frac{d}{dx}\left(e^{-5x} y\right) = 3$$

$$e^{-5x} y = 3x + C$$

$$y = 3xe^{5x} + Ce^{5x}$$

$y(0) = 0 \Rightarrow \quad 0 = 3 \cdot 0 \cdot e^{0} + Ce^{0} = C$

so $\boxed{y = 3xe^{5x}}$

4. (20 points, 5 points per part) In all parts, the unknown function is $y(x)$, and your final answer must be real-valued (not complex).

(a) Find the general solution of

$$y'' - 10y' + 25y = 0$$

$$r^2 - 10r + 25 = 0$$

$$(r-5)^2 = 0$$

repeated root $r = 5$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

(b) Find the general solution of

$$y'' + 9y = x$$

Particular solution: $y_{trial} = Ax + B$

$$y'' + 9y = 0 + 9(Ax+B) = x$$

$$\Rightarrow A = \frac{1}{9} \text{ and } B = 0$$

$$y_p = \frac{1}{9}x$$

homogeneous equation $y'' + 9y = 0$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

$$\boxed{y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{9}x}$$

(c) Find the general solution of

$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\boxed{y = C_1 e^{-x/2} \cos \frac{\sqrt{3}x}{2} + C_2 e^{-x/2} \sin \frac{\sqrt{3}x}{2}}$$

(d) Find a particular solution of

$$y'' + 4y' + 3y = e^{-x}$$

$$r^2 + 4r + 3 = 0$$

$$(r+1)(r+3) = 0$$

Since $r = -1$ is a root there is resonance.

$$y_{trial} = Axe^{-x} + Be^{-x}$$

$$y'_{trial} = A(-xe^{-x} + e^{-x}) - Be^{-x}$$

$$y''_{trial} = A(xe^{-x} - e^{-x} - e^{-x}) + Be^{-x}$$

$$y'' + 4y' + 3y = A(xe^{-x} - 2e^{-x}) + Be^{-x} + 4A(e^{-x} - xe^{-x}) - 4Be^{-x} + 3Axe^{-x} + 3Be^{-x}$$

$$= (A - 4A + 3A)\, xe^{-x} + (-2A + B + 4A - 4B + 3B)e^{-x}$$

$$= 0xe^{-x} + (2A)e^{-x}$$

we want this to equal e^{-x} so we take $A = \frac{1}{2}$, and B = anything.

So $\boxed{y_P = \frac{1}{2}xe^{-x}}$ is a particular solution.

5. (20 points) Find the Fourier series of the function $f(t)$ which is periodic with period 2 and which on the interval $-1 < t \le 1$ is defined by

$$f(t) = \begin{cases} 0 & -1 < t \le 0 \\ 5t & 0 < t \le 1 \end{cases}$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = \int_0^1 5t\,dt = \left[\frac{5}{2}t^2\right]_0^1 = \frac{5}{2}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos n\pi t\,dt = \int_0^1 5t\cos n\pi t\,dt$$

$$= \left[\frac{1}{n\pi}5t\sin n\pi t + \frac{1}{(n\pi)^2}5\cos n\pi t\right]_0^1 = \frac{5}{(n\pi)^2}(\cos n\pi - 1)$$

$$= \frac{5}{(n\pi)^2}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{-10}{(n\pi)^2} & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin n\pi t\,dt = \int_0^1 5t\sin n\pi t\,dt$$

$$= \left[-\frac{1}{n\pi}5t\cos n\pi t + \frac{1}{(n\pi)^2}5\sin n\pi t\right]_0^1$$

$$= -\frac{1}{n\pi}5\cos n\pi = \frac{(-1)5(-1)^n}{n\pi} = \frac{5(-1)^{n+1}}{n\pi}$$

$$f(t) \sim \frac{5}{4} + \sum_{n=1}^{\infty}\left(\frac{5}{(n\pi)^2}\left((-1)^n - 1\right)\cos n\pi t + \frac{5(-1)^{n+1}}{n\pi}\sin n\pi t\right)$$

6. (20 points) The equation

$$2x'' + 10x = F(t)$$

describes an undamped forced oscillator with mass $m = 2$ and spring constant $k = 10$ is driven by a driving force $F(t)$. Suppose that the position function $x(t)$ is known and is described by the Fourier series

$$x(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin \frac{n\pi t}{6}$$

Determine the driving force $F(t)$.

$$x' = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{n\pi}{6} \cos \frac{n\pi t}{6} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\frac{\pi}{6}\right) \cos \frac{n\pi t}{6}$$

$$x'' = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left(\frac{\pi}{6}\right)\left(-\frac{n\pi}{6} \sin \frac{n\pi t}{6}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\pi}{6}\right)^2 \sin \frac{n\pi t}{6}$$

$$F(t) = 2x'' + 10x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\pi}{6}\right)^2 \sin \frac{n\pi t}{6} + 10\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin \frac{n\pi t}{6}$$

$$\boxed{F(t) = \sum_{n=1}^{\infty} \left[2\left(\frac{\pi}{6}\right)^2 \frac{(-1)^{n+1}}{n} + 10 \frac{(-1)^n}{n^3} \right] \sin \frac{n\pi t}{6}}$$

7. (20 points) Consider the heat equation in a rod of length 10 ($0 \le x \le 10$)

$$\frac{\partial u}{\partial t} = 100 \frac{\partial^2 u}{\partial x^2}$$

We impose boundary conditions that the ends $x = 0$ and $x = 10$ are held fixed at zero temperature:

$$u(0,t) = 0, \quad u(10,t) = 0$$

We also impose the initial condition

$$u(x,0) = 1.$$

Find the solution $u(x,t)$.

General solution for fixed ends

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

With $k = 100$, $L = 10$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-100\left(\frac{n\pi}{10}\right)^2 t} \sin \frac{n\pi x}{10} = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin \frac{n\pi x}{10}$$

$$1 = u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{10} \quad \text{sine series}$$

$$c_n = \frac{2}{10}\int_0^{10} 1 \sin \frac{n\pi x}{10}\, dx = \frac{2}{10}\left[-\frac{10}{n\pi}\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= -\frac{2}{n\pi}(\cos n\pi - 1) = \frac{2}{n\pi}(1-(-1)^n)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\, e^{-n^2\pi^2 t} \sin \frac{n\pi x}{10}}$$

8. (20 points) Consider the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = 0 \\ y(\pi) = 0 \end{cases}$$

Show directly (that is, *without* quoting the main theorem of Sturm-Liouville theory) that there are no negative eigenvalues in this problem.

$\underline{\lambda < 0}$: Write $\lambda = -\alpha^2$ with $\alpha > 0$.

$y'' - \alpha^2 y = 0 \Rightarrow r = \pm\alpha$ so $y = Ae^{\alpha x} + Be^{-\alpha x}$

$y' = A\alpha e^{\alpha x} - B\alpha e^{-\alpha x}$

$y'(0) = 0 \Rightarrow A\alpha - B\alpha = 0 \Rightarrow A - B = 0 \Rightarrow A = B$

Thus $y(x) = A(e^{\alpha x} + e^{-\alpha x})$

$y(\pi) = 0 \Rightarrow A(e^{\alpha\pi} + e^{-\alpha\pi}) = 0$

This forces $A = 0$ unless $e^{\alpha\pi} + e^{-\alpha\pi} = 0$

$e^{\alpha\pi} = -e^{-\alpha\pi}$

$e^{2\alpha\pi} = -1$

But $e^{2\alpha\pi}$ is always positive, so this is impossible.

Thus A is forced to be zero, and $\lambda = -\alpha^2 < 0$ is not an eigenvalue.

9. (20 points) Let $X(x) = e^{10x} + e^{-10x}$. Find a nonzero function $Y(y)$ such that the product

$$u(x, y) = X(x)Y(y)$$

satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If $u(x,y) = X(x)Y(y) = (e^{10x} + e^{-10x})Y(y)$

Then $\frac{\partial u}{\partial x} = (10e^{10x} - 10e^{-10x})Y(y)$

$$\frac{\partial^2 u}{\partial x^2} = (100e^{10x} + 100e^{-10x})Y(y)$$

$$\frac{\partial u}{\partial y} = (e^{10x} + e^{-10x})\frac{dY}{dy}$$

$$\frac{\partial^2 u}{\partial y^2} = (e^{10x} + e^{-10x})\frac{d^2Y}{dy^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 100(e^{10x} + e^{-10x})Y(y) + (e^{10x} + e^{-10x})\frac{d^2Y}{dy^2}$$

For this to be zero we must have

$$\frac{d^2Y}{dy^2} + 100Y = 0$$

General solution $\boxed{Y(y) = A\cos 10y + B\sin 10y}$

Also correct: $Y(y) = \cos 10y$

$Y(y) = \sin 10y$

or any particular combination thereof.

10. (20 points) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $0 \le x \le 10$. We impose the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(10,t) = 0.$$

The general solution of the wave equation with these conditions may be found by separation of variables. The result is:

$$u(x,t) = C_0 + C_1 t + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi t}{10} + B_n \sin\frac{n\pi t}{10} \right) \cos\frac{n\pi x}{10}$$

where C_0, C_1, A_n, B_n are constants. **You are not being asked to derive this formula; you should take it as given in this problem.**

Your task: Find the function $u(x,t)$ that satisfies the wave equation and boundary conditions described above as well as the initial conditions

$$u(x,0) = 1, \quad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{2}{n^2} \cos\frac{n\pi x}{10}.$$

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{10} = 1$$

Take $C_0 = 1$, $A_n = 0$

$$\frac{\partial u}{\partial t}(x,0) = C_1 + \sum_{n=1}^{\infty} \frac{n\pi}{10} B_n \cos\frac{n\pi x}{10} = \sum_{n=1}^{\infty} \frac{2}{n^2} \cos\frac{n\pi x}{10}$$

Take $C_1 = 0$, $\frac{n\pi}{10} B_n = \frac{2}{n^2} \Rightarrow B_n = \frac{20}{n^3\pi}$

$$\boxed{u(x,t) = 1 + \sum_{n=1}^{\infty} \frac{20}{n^3\pi} \sin\frac{n\pi t}{10} \cos\frac{n\pi x}{10}}$$

11. **WARNING: THIS EXTRA CREDIT PROBLEM IS DIFFICULT and should only be attempted after you have completed the rest of the exam to your satisfaction. Partial credit is available on this problem.**

    (20 points Extra Credit) Show that the locus of points (x, y) in the plane satisfying the equation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin nx \sin ny = 0$$

    consists of two sets of lines dividing the plane into squares of area π^2.

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