EXTRA CREDIT SOLUTION

JAMES PASCALEFF

Problem.[1] (20 points) Show that the locus of points (x, y) in the plane satisfying the equation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin nx \sin ny = 0$$

consists of two sets of lines dividing the plane into squares of area π^2.

Remarks. It is clear that if either $x = k\pi$ or $y = k\pi$ for an integer k, then all terms in the series are equal to zero. The equations $x = k\pi$ define vertical lines, while the equations $y = k\pi$ define horizontal lines. These lines divide the plane into squares of side length π. I gave 5 points for figuring this out. The real matter of the problem is to show that the function

$$F(x, y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin nx \sin ny$$

is *not* equal to zero at any points other than those where $x = k\pi$ or $y = k\pi$.

Solution. Using the identity

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

we find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin nx \sin ny = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \frac{1}{2}[\cos(n(x - y)) - \cos(n(x + y))]$$

Denoting by $F(x, y)$ this function, and defining a new function $G(z)$ by

$$G(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n^2} \cos nz$$

we have the identity[2]

$$F(x, y) = G(x - y) - G(x + y)$$

We now seek to determine the function $G(z)$. Some guesswork is involved at this point. The fact that $G(z)$ is even, together with the n^2 in the denominator, suggest that $G(z)$ is related to z^2. To that end, let $g(z)$ be the periodic function of period 2π that on the interval $-\pi \leq z < \pi$ satisfies

[1] According to *A Course of Modern Analysis* by Whittaker and Watson, this problem appeared on the Cambridge Mathematical Tripos in the year 1895.

[2] The function $F(x, y)$ satisfies the wave equation $F_{xx} = F_{yy}$, and this identity is the decomposition into left-moving and right-moving waves.

$g(z) = z^2$. The Fourier series of this function has only a constant term and cosine terms. The coefficients are

$$a_0 = \frac{2}{\pi}\int_0^\pi z^2\,dz = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi z^2 \cos nz\,dz = \frac{2}{\pi}\left[\frac{1}{n}z^2\sin nz + \frac{2}{n^2}z\cos nz - \frac{2}{n^3}\sin nz\right]_0^\pi$$

$$= \frac{2}{\pi}\frac{2}{n^2}\pi\cos n\pi = \frac{4(-1)^n}{n^2}$$

$$g(z) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nz$$

Comparing with $G(z)$, we find $(g(z) - \pi^2/3)/4 = -2G(z)$, or

$$G(z) = \frac{\pi^2}{24} - \frac{1}{8}g(z)$$

With this in hand,

$$F(x,y) = G(x-y) - G(x+y) = \left(\frac{\pi^2}{24} - \frac{1}{8}g(x-y)\right) - \left(\frac{\pi^2}{24} - \frac{1}{8}g(x+y)\right) = \frac{1}{8}(g(x+y) - g(x-y))$$

The locus defined by the equation $F(x,y) = 0$ is thus the same as that defined by the equation $g(x+y) = g(x-y)$. Change variables to $u = x+y$ and $v = x-y$, so $x = (u+v)/2$ and $y = (u-v)/2$. We must solve the equation $g(u) = g(v)$. As $g(z) = z^2$ for $-\pi < z < \pi$, and $g(z)$ is periodic of period 2π, the only way $g(u)$ can equal $g(v)$ is if either $u = v + 2\pi k$ or $u = -v + 2\pi k$ for some integer k.[3] The condition $u = v + 2\pi k$ is equivalent to $y = (u-v)/2 = \pi k$, while the condition $u = -v + 2\pi k$ is equivalent to $x = (u+v)/2 = \pi k$. Thus we have shown that $F(x,y) = 0$ if and only if either x or y is an integer multiple of π.

[3] This is more generally true as long as $g(z)$ is a 2π-periodic even function that is strictly monotonic on the interval $0 \le z \le \pi$.