NAME: Solutions

NetID:

MATH 285 G1 Exam 3 (B) **April 18, 2016** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, notes, or calculators.

Orthogonality formulas

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{1}$$

$$\int_{-L}^{L} \sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2}$$

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = 0 \tag{3}$$

Some integral formulas

$$\int u\cos u\,du = u\sin u + \cos u + C \tag{4}$$

$$\int u\sin u\,du = -u\cos u + \sin u + C \tag{5}$$

1. (20 points) An undamped oscillator with mass $m = 3$ and spring constant $k = 9$ is driven by a driving force $F(t)$ which is given as a Fourier series

$$F(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi t + \frac{2}{n} \sin n\pi t$$

The differential equation for $x(t)$ is

$$3x'' + 9x = F(t)$$

Find a particular solution of this equation.

$$x(t) = \sum_{n=1}^{\infty} A_n \cos n\pi t + B_n \sin n\pi t$$

$$3x'' + 9x = \sum_{n=1}^{\infty} (-3(n\pi)^2 + 9) A_n \cos n\pi t + (-3(n\pi)^2 + 9) B_n \sin n\pi t$$

This should $= \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi t + \frac{2}{n} \sin n\pi t$

so $(-3(n\pi)^2 + 9) A_n = \frac{1}{n^2}$, $A_n = \frac{1}{n^2(9-3(n\pi)^2)}$

$(-3(n\pi)^2 + 9) B_n = \frac{2}{n}$, $B_n = \frac{2}{n(9-3(n\pi)^2)}$

$$x(t) = \sum_{n=1}^{\infty} \frac{1}{n^2(9-3(n\pi)^2)} \cos n\pi t + \frac{2}{n(9-3(n\pi)^2)} \sin n\pi t$$

2. (a) (10 points) Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{4n + n^2} \cos nt$$

Use the orthogonality formulas to evaluate the integral

$$\int_{-\pi}^{\pi} f(t) \cos 3t \, dt$$

$$= \sum_{n=1}^{\infty} \frac{1}{4n+n^2} \int_{-\pi}^{\pi} \cos nt \cos 3t \, dt$$

} by orthogonality ($L = \pi$) all terms are zero except $n = 3$

$$= \frac{1}{4 \cdot 3 + 3^2} \int_{-\pi}^{\pi} \cos 3t \cos 3t \, dt = \frac{\pi}{4 \cdot 3 + 3^2} = \frac{\pi}{12+9} = \boxed{\frac{\pi}{21}}$$

(b) (10 points) Let $g(t)$ be the function which is periodic of period 16, and which is defined on the interval $-8 \le t < 8$ by the formula

$$g(t) = 3t + 2t^2 + 4\cos(t)$$

Set up, but do not evaluate, an integral expression for the coefficient of $\cos \frac{3\pi t}{8}$ in the Fourier series of $g(t)$ (also known as a_3 in our standard notation).

$$\boxed{a_3 = \frac{1}{8} \int_{-8}^{8} \left(3t + 2t^2 + 4\cos t\right) \cos \frac{3\pi t}{8} \, dt}$$

3. (a) (5 points) Consider the function which is periodic of period 2π defined on the interval $-\pi \le t < \pi$

$$f(t) = \begin{cases} 22, & -\pi \le t < 0 \\ 2875/\pi^2, & t = 0 \\ 10, & 0 < t < \pi \end{cases}$$

If we take the Fourier series of $f(t)$, and put $t = 0$ in that series, what number does it converge to? Put another way, what is the sum of the Fourier series of $f(t)$ at $t = 0$? Explain your answer (briefly).

$t=0$ is a jump discontinuity. The Fourier series converges to the average of the one sided limits:

$$\frac{1}{2}\left[\lim_{t\to 0^-} f(t) + \lim_{t\to 0^+} f(t)\right] = \frac{1}{2}\left[22+10\right] = \boxed{16}$$

(b) (15 points) Consider the function defined by the Fourier series

$$g(t) = \sum_{n=1}^{\infty} 4e^{-2\pi n} \cos n\pi t$$

Find a Fourier series expression for the antiderivative $\int g(t)\,dt$. You are *not* expected to address the question of convergence.

$$\int g(t)\,dt = \sum_{n=1}^{\infty} 4e^{-2\pi n} \int \cos n\pi t \, dt$$

$$= \sum_{n=1}^{\infty} 4e^{-2\pi n} \frac{1}{n\pi} \sin n\pi t + C$$

$$= \boxed{C + \sum_{n=1}^{\infty} \frac{4e^{-2\pi n}}{n\pi} \sin n\pi t}$$

C is the constant of integration

4. (20 points) Let $f(t)$ be an even periodic function of period 4 such that, on the interval $0 < t < 2$,

$$f(t) = -4t, \quad 0 \leq t < 2$$

Find the Fourier series of $f(t)$.

$\underline{L = 2:}$

Since $f(t)$ is even, $b_n = 0$, and

$$a_0 = \frac{2}{2}\int_0^2 f(t)\,dt, \quad a_n = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}\,dt$$

$$a_0 = \int_0^2 -4t\,dt = \left[-2t^2\right]_0^2 = -8$$

$$a_n = \int_0^2 -4t\cos\frac{n\pi t}{2}\,dt = \left[-4t\sin\frac{n\pi t}{2}\right]_0^2 - \int_0^2 \frac{2}{n\pi}\sin\frac{n\pi t}{2}\cdot(-4)\,dt$$

$$\left(\begin{matrix} u = -4t & du = -4dt \\ dv = \cos\frac{n\pi t}{2}\,dt & v = \frac{2}{n\pi}\sin\frac{n\pi t}{2} \end{matrix}\right)$$

$$= -8\sin n\pi - 0 + \frac{8}{n\pi}\left[-\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_0^2$$

$$= 0 - \frac{16}{(n\pi)^2}\left(\cos n\pi - \cos 0\right) = \frac{-16}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$$

$$\boxed{f(t) = -4 + \sum_{n=1}^{\infty} \frac{16}{(n\pi)^2}\left(1 - (-1)^n\right)\cos\frac{n\pi t}{2}}$$

5. (20 points) Consider the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y'(10) = 0 \end{cases}$$

Find the eigenvalues, and find a single nonzero eigenfunction associated to each eigenvalue. You may assume that all the eigenvalues are positive, for indeed they are.

Assume $\lambda > 0$: $y'' + \lambda y = 0$ implies

$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$y'(x) = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x$

$y(0)=0$ implies $A\cos 0 + B\sin 0 = 0$, $A = 0$

Thus $y(x) = B\sin\sqrt{\lambda}x$

The condition $y'(10)=0$ implies $\sqrt{\lambda}B\cos\sqrt{\lambda}\cdot 10 = 0$

this will force B to be zero unless $\cos 10\sqrt{\lambda} = 0$.

The condition $\cos 10\sqrt{\lambda} = 0$ means that $10\sqrt{\lambda}$ is an odd multiple of $\frac{\pi}{2}$. $10\sqrt{\lambda} = \frac{(2n-1)\pi}{2}$ $n = 1, 2, 3, \ldots$

$\sqrt{\lambda} = \frac{(2n-1)\pi}{20}$, $\lambda = \left(\frac{(2n-1)\pi}{20}\right)^2$

The eigenvalues are $\lambda_n = \left(\frac{(2n-1)\pi}{20}\right)^2$, $n = 1, 2, 3, \ldots$

eigenfunctions are $y_n(x) = \sin\frac{(2n-1)\pi x}{20}$, $n = 1, 2, 3, \ldots$

This page is for work that doesn't fit on other pages.