NAME: Solutions

NetID:

MATH 285 G1 Exam 3 (A) **April 18, 2016** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, notes, or calculators.

Orthogonality formulas

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\cos\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{1}$$

$$\int_{-L}^{L} \sin\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{2}$$

$$\int_{-L}^{L} \cos\frac{m\pi t}{L}\sin\frac{n\pi t}{L}\,dt = 0 \tag{3}$$

Some integral formulas

$$\int u\cos u\,du = u\sin u + \cos u + C \tag{4}$$

$$\int u\sin u\,du = -u\cos u + \sin u + C \tag{5}$$

1. (20 points) An undamped oscillator with mass $m = 2$ and spring constant $k = 10$ is driven by a driving force $F(t)$ which is given as a Fourier series

$$F(t) = \sum_{n=1}^{\infty} \frac{5}{n^2} \cos n\pi t + \frac{1}{n^3} \sin n\pi t$$

The differential equation for $x(t)$ is

$$2x'' + 10x = F(t)$$

Find a particular solution of this equation.

$$x(t) = \sum_{n=1}^{\infty} A_n \cos n\pi t + B_n \sin n\pi t$$

$$2x'' + 10x = \sum_{n=1}^{\infty} (-2(n\pi)^2 + 10) A_n \cos n\pi t + (-2(n\pi)^2 + 10) B_n \sin n\pi t$$

This should $= \sum_{n=1}^{\infty} \frac{5}{n^2} \cos n\pi t + \frac{1}{n^3} \sin n\pi t$

so $(-2(n\pi)^2 + 10) A_n = \frac{5}{n^2}$, $A_n = \frac{5}{n^2(10 - 2(n\pi)^2)}$

$(-2(n\pi)^2 + 10) B_n = \frac{1}{n^3}$, $B_n = \frac{1}{n^3(10 - 2(n\pi)^2)}$

$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{5}{n^2(10 - 2(n\pi)^2)} \cos n\pi t + \frac{1}{n^3(10 - 2(n\pi)^2)} \sin n\pi t}$$

2. (a) (10 points) Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 3n} \cos nt$$

Use the orthogonality formulas to evaluate the integral

$$\int_{-\pi}^{\pi} f(t) \cos 4t \, dt$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+3n} \int_{-\pi}^{\pi} \cos nt \cos 4t \, dt$$

by orthogonality ($L = \pi$) all terms are zero except $n=4$

$$= \frac{(-1)^4}{4^2+3\cdot 4} \int_{-\pi}^{\pi} \cos 4t \cos 4t \, dt = \frac{(-1)^4 \pi}{4^2+3\cdot 4} = \frac{\pi}{16+12} = \boxed{\frac{\pi}{28}}$$

(b) (10 points) Let $g(t)$ be the function which is periodic of period 20, and which is defined on the interval $-10 \leq t < 10$ by the formula

$$g(t) = 3t^2 + e^t + 4$$

Set up, but do not evaluate, an integral expression for the coefficient of $\sin \frac{3\pi t}{10}$ in the Fourier series of $g(t)$ (also known as b_3 in our standard notation).

$$\boxed{b_3 = \frac{1}{10} \int_{-10}^{10} (3t^2 + e^t + 4) \sin \frac{3\pi t}{10} \, dt}$$

3. (a) (5 points) Consider the function which is periodic of period 2π defined on the interval $-\pi \le t < \pi$

$$f(t) = \begin{cases} 3, & -\pi \le t < 0 \\ e^{\pi^2}, & t = 0 \\ -1, & 0 < t < \pi \end{cases}$$

If we take the Fourier series of $f(t)$, and put $t = 0$ in that series, what number does it converge to? Put another way, what is the sum of the Fourier series of $f(t)$ at $t = 0$? Explain your answer (briefly).

$t=0$ is a jump discontinuity. The Fourier series converges to the average of the one sided limits:

$$\frac{1}{2}\left[\lim_{t\to 0^-} f(t) + \lim_{t\to 0^+} f(t)\right] = \frac{1}{2}\left[3 + -1\right] = \boxed{1}$$

(b) (15 points) Consider the function defined by the Fourier series

$$g(t) = \sum_{n=1}^{\infty} \frac{3e^{-n\pi}}{n^2} \cos n\pi t$$

Find a Fourier series expression for the antiderivative $\int g(t)\,dt$. You are *not* expected to address the question of convergence.

$$\int g(t)\,dt = \sum_{n=1}^{\infty} \frac{3e^{-n\pi}}{n^2} \int \cos n\pi t \, dt$$

$$= \sum_{n=1}^{\infty} \frac{3e^{-n\pi}}{n^2} \frac{1}{n\pi} \sin n\pi t + C$$

$$= \boxed{C + \sum_{n=1}^{\infty} \frac{3e^{-n\pi}}{n^3\pi} \sin n\pi t}$$

C is the constant of integration

4. (20 points) Let $f(t)$ be an even periodic function of period 4 such that, on the interval $0 < t < 2$,

$$f(t) = 2t, \quad 0 \le t < 2$$

Find the Fourier series of $f(t)$.

$\underline{L = 2:}$

Since $f(t)$ is even, $b_n = 0$, and

$$a_0 = \frac{2}{2}\int_0^2 f(t)\,dt \quad , \quad a_n = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}\,dt$$

$$a_0 = \int_0^2 2t\,dt = \left[t^2\right]_0^2 = 4$$

$$a_n = \int_0^2 2t\cos\frac{n\pi t}{2}\,dt = \left[2t\sin\frac{n\pi t}{2}\right]_0^2 - \int_0^2 \frac{2}{n\pi}\sin\frac{n\pi t}{2}\cdot 2\,dt$$

$$\left(\begin{array}{ll} u = 2t & du = 2\,dt \\ dv = \cos\frac{n\pi t}{2}\,dt & v = \frac{2}{n\pi}\sin\frac{n\pi t}{2} \end{array}\right)$$

$$= 4\sin n\pi - 0 - \frac{4}{n\pi}\left[-\frac{2}{n\pi}\cos\frac{n\pi t}{2}\right]_0^2$$

$$= 0 + \frac{8}{(n\pi)^2}\left(\cos n\pi - \cos 0\right) = \frac{8}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$$

$$\boxed{f(t) = 2 + \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2}\left((-1)^n - 1\right)\cos\frac{n\pi t}{2}}$$

5. (20 points) Consider the eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = 0 \\ y(20) = 0 \end{cases}$$

Find the eigenvalues, and find a single nonzero eigenfunction associated to each eigenvalue. You may assume that all the eigenvalues are positive, for indeed they are.

Assume $\lambda > 0$: $y'' + \lambda y = 0$ implies

$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$y'(x) = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + \sqrt{\lambda}B\cos\sqrt{\lambda}x$

$y'(0)=0$ implies $-\sqrt{\lambda}A\sin 0 + \sqrt{\lambda}B\cos 0 = 0$,

$\sqrt{\lambda}B = 0$, $B = 0$.

Thus $y(x) = A\cos\sqrt{\lambda}x$

The condition $y(20) = 0$ implies $A\cos 20\sqrt{\lambda} = 0$.

this will force A to be zero unless $\cos 20\sqrt{\lambda} = 0$.

The condition $\cos 20\sqrt{\lambda} = 0$ means that $20\sqrt{\lambda}$ is an odd multiple of $\frac{\pi}{2}$. $20\sqrt{\lambda} = \frac{(2n-1)\pi}{2}$ $n = 1, 2, 3, \ldots$

$\sqrt{\lambda} = \frac{(2n-1)\pi}{40}$, $\lambda = \left(\frac{(2n-1)\pi}{40}\right)^2$

The eigenvalues are $\lambda_n = \left(\frac{(2n-1)\pi}{40}\right)^2$, $n = 1, 2, 3, \ldots$

eigenfunctions are $y_n(x) = \cos\frac{(2n-1)\pi x}{40}$, $n = 1, 2, 3, \ldots$

This page is for work that doesn't fit on other pages.