NAME: Solutions

NetID:

MATH 285 G1 Exam 1 (C) **February 17, 2016** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

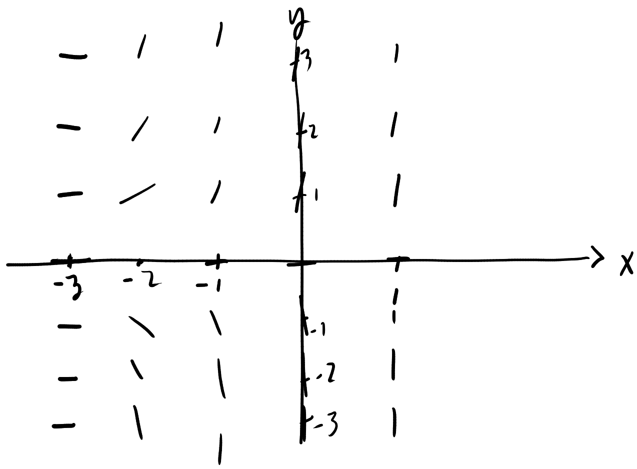
INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No notes, books, calculators, or other electronic devices are permitted.

1. (20 points) Consider the differential equation

$$\frac{dy}{dx} = (x+3)y$$

(a) (10 points) Draw a slope field for this equation.



(b) (10 points) Given the initial condition $y(1) = 1$, use Euler's method with two steps to approximate $y(1.2)$.

Step size $h = .1$

$x_0 = 1 \qquad y_0 = 1$

$x_1 = 1.1 \qquad y_1 = y_0 + h(x_0+3)y_0 = 1 + .1(4)(1) = 1.4$

$x_2 = 1.2 \qquad y_2 = y_1 + h(x_1+3)y_1 = 1.4 + .1(4.1)(1.4)$

$= 1.4 + .1(5.74)$

$= 1.4 + .574$

$= 1.974$

$$\begin{array}{r} 41 \\ \times 14 \\ \hline 164 \\ 410 \\ \hline 574 \end{array}$$

so $\boxed{y(1.2) \approx 1.974}$

2. (20 points) Let $y(x)$ be a solution of the initial value problem

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 1$$

Starting from $y_0(x) = 1$, compute the first and second Picard approximations $y_1(x)$ and $y_2(x)$, and use $y_2(x)$ to estimate $y(0.1)$.

$$y_1(x) = 1 + \int_0^x \left(1 + y_0(t)^2\right) dt = 1 + \int_0^x (1 + 1^2)\, dt$$

$$= 1 + \int_0^x 2\, dt = 1 + 2x$$

$$y_2(x) = 1 + \int_0^x \left(1 + y_1(t)^2\right) dt = 1 + \int_0^x \left(1 + (1 + 2t)^2\right) dt$$

$$= 1 + \int_0^x (1 + 1 + 4t + 4t^2)\, dt$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$y(0.1) \approx y_2(0.1) = 1 + 2(.1) + 2(.1)^2 + \frac{4}{3}(.1)^3$$

$$\approx 1 + .2 + .02 + 1.3\ (.1)^3$$

$$= 1.22 + .0013 \quad = 1.2213$$

3. (20 points) Find the general solution of

$$\frac{dy}{dx} = e^{5x} - 2y$$

First order linear equation:

$$\frac{dy}{dx} + 2y = e^{5x}$$

$$P(x) = 2 \qquad Q(x) = e^{5x}$$

Integrating factor $\rho(x) = e^{\int P(x)\,dx} = e^{2x}$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}e^{5x} = e^{7x}$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{7x}$$

$$e^{2x}y = \int e^{7x}\,dx = \frac{1}{7}e^{7x} + C$$

$$\boxed{y = \frac{1}{7}e^{5x} + Ce^{-2x}}$$

4. (20 points) Let $P(t)$ be denote a population of fish in a lake. This population is governed by the differential equation

$$\frac{dP}{dt} = P(300 - P) - 500$$

(a) (10 points) Find the equilibrium solutions, and determine whether each is stable or unstable.

Equilibrium solutions are solutions of

$$P(300-P) - 500 = 0$$

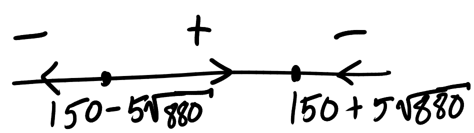
$$-P^2 + 300P - 500 = 0$$

$$P^2 - 300P + 500 = 0$$

$$P = \frac{300 \pm \sqrt{300^2 - 4 \cdot 500}}{2} = \frac{300 \pm \sqrt{90000 - 2000}}{2} = \frac{300 \pm \sqrt{88000}}{2}$$

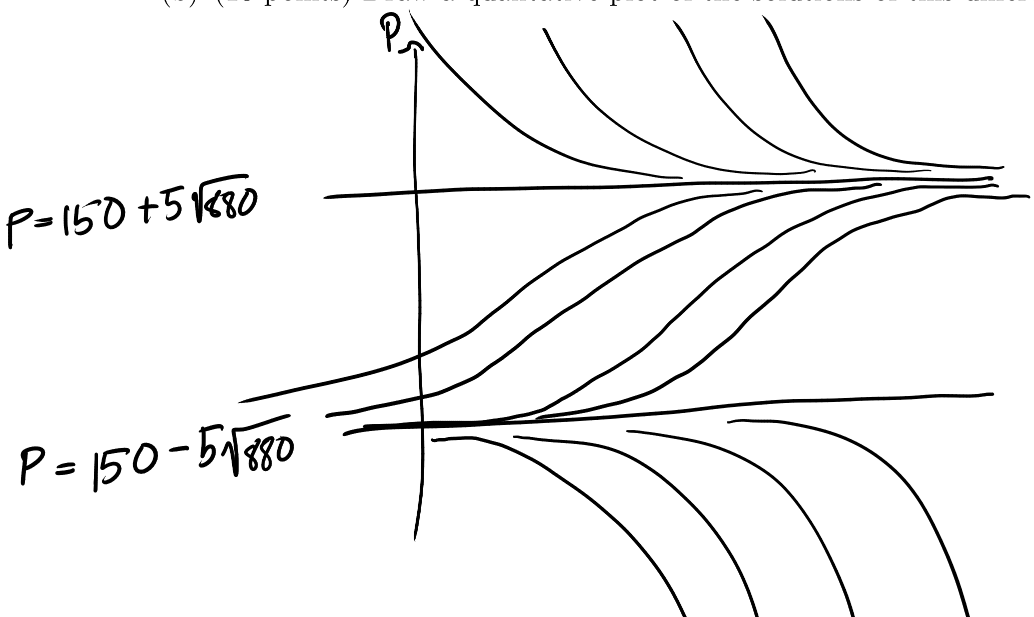
$$= 150 \pm 5\sqrt{880}$$

$$f(P) = -P^2 + 300P - 500$$



$$\boxed{\begin{array}{l} P = 150 + 5\sqrt{880} \text{ is } \underline{\text{stable}} \\ P = 150 - 5\sqrt{880} \text{ is } \underline{\text{unstable}}. \end{array}}$$

(b) (10 points) Draw a qualitative plot of the solutions of this differential equation.



5. (20 points) A metal ball has been heated to $400°C$. It is placed into a bath of water at $30°C$. After 5 seconds, it has cooled to a temperature of $200°C$.

Suppose now that the metal ball is cooled to $0°C$, and again placed into a bath of water at $30°C$. How long will it take to reach a temperature of $20°C$? Your answer does not need to be simplified.

In both situations, the process is governed by Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - A)$$

where A is the temperature of the water, and k is a constant.

$$\frac{dT}{dt} = -k(T-A)$$

$$\int \frac{dT}{T-A} = \int -k\,dt$$

$$\ln|T-A| = -kt + C$$

$$|T-A| = e^{-kt} e^{C}$$

$$T-A = Ce^{-kt}$$

$$T = A + Ce^{-kt}$$

First case $A=30$

$$T(0) = 400$$

$$T(5) = 200$$

$$400 = 30 + Ce^{-k0}$$

$$370 = C$$

$$200 = 30 + 370e^{-k5}$$

$$170 = 370e^{-5k}$$

$$e^{5k} = \frac{370}{170} = \frac{37}{17}$$

$$\boxed{k = \frac{1}{5}\ln\left(\frac{37}{17}\right)}$$

Second case $A = 30$

$$T(0) = 0$$

$$0 = 30 + Ce^{0}$$

$$-30 = C$$

$$T(t) = 30 - 30e^{-kt}$$

$$T(t) = 20$$

$$20 = 30 - 30e^{-kt}$$

$$30e^{-kt} = 10$$

$$e^{-kt} = \frac{1}{3}$$

$$-kt = \ln\left(\frac{1}{3}\right) = -\ln(3)$$

$$t = \frac{\ln(3)}{k}$$

$$\boxed{t = \frac{\ln(3)}{\frac{1}{5}\ln\left(\frac{37}{17}\right)}}$$

This page is for work that doesn't fit on the other pages.