NAME: Solutions

NetID:

MATH 285 G1 Exam 1 (B) **February 17, 2016** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

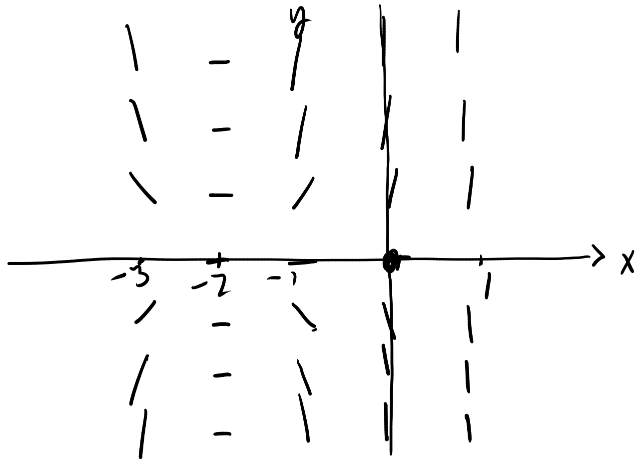
INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No notes, books, calculators, or other electronic devices are permitted.

1. (20 points) Consider the differential equation

$$\frac{dy}{dx} = (x+2)y$$

(a) (10 points) Draw a slope field for this equation.



(b) (10 points) Given the initial condition $y(1) = 1$, use Euler's method with two steps to approximate $y(1.2)$.

Step size $h = .1$

$x_0 = 1 \qquad y_0 = 1$

$x_1 = 1.1 \qquad y_1 = y_0 + h(x_0+2)y_0 = 1 + .1(3)(1) = 1.3$

$x_2 = 1.2 \qquad y_2 = y_1 + h(x_1+2)y_1 = 1.3 + .1(3.1)(1.3)$

$= 1.3 + .1(4.03)$

$= 1.3 + .403$

$= 1.703$

$$\begin{array}{r} 31 \\ \times 13 \\ \hline 93 \\ 310 \\ \hline 403 \end{array}$$

so $\boxed{y(1.2) \approx 1.703}$

2. (20 points) Let $y(x)$ be a solution of the initial value problem

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 2$$

Starting from $y_0(x) = 2$, compute the first and second Picard approximations $y_1(x)$ and $y_2(x)$, and use $y_2(x)$ to estimate $y(0.1)$.

$$y_1(x) = 2 + \int_0^x \left(1 + y_0(t)^2\right) dt = 2 + \int_0^x (1+2^2)\, dt$$

$$= 2 + \int_0^x 5\, dt = 2 + 5x$$

$$y_2(x) = 2 + \int_0^x \left(1 + y_1(t)^2\right) dt = 2 + \int_0^x \left(1 + (2+5t)^2\right) dt$$

$$= 2 + \int_0^x (1 + 4 + 20t + 25t^2)\, dt$$

$$= 2 + 5x + 10x^2 + \frac{25}{3}x^3$$

$$y(0.1) \approx y_2(0.1) = 2 + 5(.1) + 10(.1)^2 + \frac{25}{3}(.1)^3$$

$$\approx 2 + .5 + .1 + 8.3(.1)^3$$

$$= 2.6 + .0083$$

$$= 2.6083$$

3. (20 points) Find the general solution of

$$\frac{dy}{dx} = e^{3x} - 4y$$

First order linear equation:

$$\frac{dy}{dx} + 4y = e^{3x}$$

$$P(x) = 4 \qquad Q(x) = e^{3x}$$

Integrating factor $\rho(x) = e^{\int P(x)\,dx} = e^{4x}$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = e^{4x}e^{3x} = e^{7x}$$

$$\frac{d}{dx}\left(e^{4x}y\right) = e^{7x}$$

$$e^{4x}y = \int e^{7x}\,dx = \frac{1}{7}e^{7x} + C$$

$$\boxed{y = \frac{1}{7}e^{3x} + Ce^{-4x}}$$

4. (20 points) Let $P(t)$ be denote a population of fish in a lake. This population is governed by the differential equation

$$\frac{dP}{dt} = P(200 - P) - 100$$

(a) (10 points) Find the equilibrium solutions, and determine whether each is stable or unstable.

Equilibrium solutions are solutions of

$$P(200-P) - 100 = 0$$

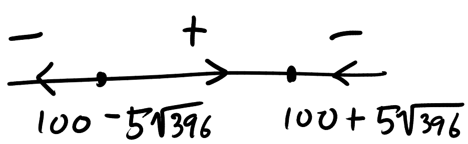
$$-P^2 + 200P - 100 = 0$$

$$P^2 - 200P + 100 = 0$$

$$P = \frac{200 \pm \sqrt{200^2 - 4\cdot 100}}{2} = \frac{200 \pm \sqrt{40000 - 400}}{2} = \frac{200 \pm \sqrt{39600}}{2}$$

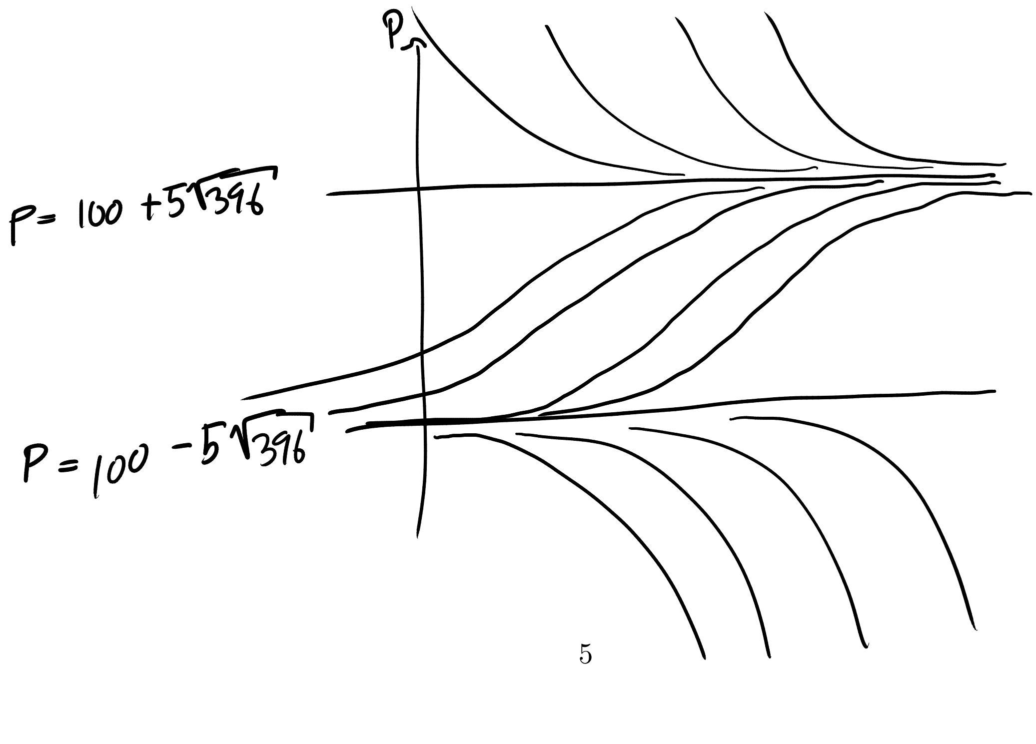
$$= 100 \pm 5\sqrt{396}$$

$$f(P) = -P^2 + 200P - 100$$



$$\boxed{\begin{array}{l} P = 100 + 5\sqrt{396} \text{ is } \underline{\text{stable}} \\ P = 100 - 5\sqrt{396} \text{ is } \underline{\text{unstable}}. \end{array}}$$

(b) (10 points) Draw a qualitative plot of the solutions of this differential equation.



5. (20 points) A metal ball has been heated to $300°C$. It is placed into a bath of water at $30°C$. After 5 seconds, it has cooled to a temperature of $200°C$.

Suppose now that the metal ball is cooled to $0°C$, and again placed into a bath of water at $30°C$. How long will it take to reach a temperature of $20°C$? Your answer does not need to be simplified.

In both situations, the process is governed by Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - A)$$

where A is the temperature of the water, and k is a constant.

$$\frac{dT}{dt} = -k(T-A)$$

$$\int \frac{dT}{T-A} = \int -k\,dt$$

$$\ln|T-A| = -kt + C$$

$$|T-A| = e^{-kt}e^{C}$$

$$T-A = Ce^{-kt}$$

$$T = A + Ce^{-kt}$$

First case $A=30$

$T(0)=300$

$T(5)=200$

$300 = 30 + Ce^{-k0}$

$270 = C$

$200 = 30 + 270e^{-k5}$

$170 = 270e^{-5k}$

$e^{5k} = \frac{270}{170} = \frac{27}{17}$

$$\boxed{k = \frac{1}{5}\ln\left(\frac{27}{17}\right)}$$

Second case $A = 30$

$T(0) = 0$

$0 = 30 + Ce^{0}$

$-30 = C$

$T(t) = 30 - 30e^{-kt}$

$T(t) = 20$

$20 = 30 - 30e^{-kt}$

$30e^{-kt} = 10$

$e^{-kt} = \frac{1}{3}$

$-kt = \ln\left(\frac{1}{3}\right) = -\ln(3)$

$t = \frac{\ln(3)}{k}$

$$\boxed{t = \frac{\ln(3)}{\frac{1}{5}\ln\left(\frac{27}{17}\right)}}$$

This page is for work that doesn't fit on the other pages.