NAME: Solutions

NetID:

MATH 285 G1 Exam 1 (A) **February 17, 2016** **Instructor: Pascaleff**

Problem	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

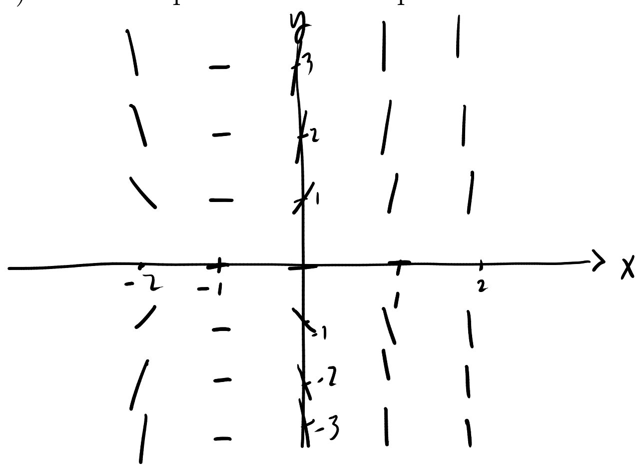
INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No notes, books, calculators, or other electronic devices are permitted.

1. (20 points) Consider the differential equation

$$\frac{dy}{dx} = (x+1)y$$

(a) (10 points) Draw a slope field for this equation.



(b) (10 points) Given the initial condition $y(1) = 1$, use Euler's method with two steps to approximate $y(1.2)$.

Step size $h = .1$

$x_0 = 1 \qquad y_0 = 1$

$x_1 = 1.1 \qquad y_1 = y_0 + h(x_0+1)y_0 = 1 + .1(2)(1) = 1.2$

$x_2 = 1.2 \qquad y_2 = y_1 + h(x_1+1)y_1 = 1.2 + .1(2.1)(1.2)$

$= 1.2 + .1(2.52)$

$= 1.2 + .252$

$= 1.452$

$$\begin{array}{r} 21 \\ \times 12 \\ \hline 42 \\ 210 \\ \hline 252 \end{array}$$

so $\boxed{y(1.2) \approx 1.452}$

2. (20 points) Let $y(x)$ be a solution of the initial value problem

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 3$$

Starting from $y_0(x) = 3$, compute the first and second Picard approximations $y_1(x)$ and $y_2(x)$, and use $y_2(x)$ to estimate $y(0.1)$.

$$y_1(x) = 3 + \int_0^x (1 + y_0(t)^2)\, dt = 3 + \int_0^x (1+3^2)\, dt$$

$$= 3 + \int_0^x 10\, dt = 3 + 10x$$

$$y_2(x) = 3 + \int_0^x (1 + y_1(t)^2)\, dt = 3 + \int_0^x (1 + (3+10t)^2)\, dt$$

$$= 3 + \int_0^x (1 + 9 + 60t + 100t^2)\, dt$$

$$= 3 + 10x + 30x^2 + \frac{100}{3} x^3$$

$$y(0.1) \approx y_2(0.1) = 3 + 10(.1) + 30(.1)^2 + \frac{100}{3}(.1)^3$$

$$= 3 + 1 + .3 + 33.3(.1)^3$$

$$= 4.3 + .0333 = 4.3333$$

3. (20 points) Find the general solution of

$$\frac{dy}{dx} = e^{2x} - 3y$$

First order linear equation:

$$\frac{dy}{dx} + 3y = e^{2x}$$

$$P(x) = 3 \qquad Q(x) = e^{2x}$$

Integrating factor $\rho(x) = e^{\int P(x)\,dx} = e^{3x}$

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}e^{2x} = e^{5x}$$

$$\frac{d}{dx}\left(e^{3x}y\right) = e^{5x}$$

$$e^{3x}y = \int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$$

$$\boxed{y = \frac{1}{5}e^{2x} + Ce^{-3x}}$$

4. (20 points) Let $P(t)$ be denote a population of fish in a lake. This population is governed by the differential equation

$$\frac{dP}{dt} = P(100 - P) - 200$$

(a) (10 points) Find the equilibrium solutions, and determine whether each is stable or unstable.

Equilibrium solutions are solutions of

$$P(100-P)-200 = 0$$

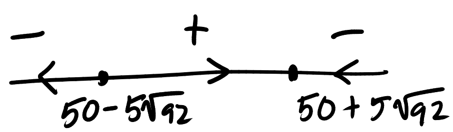
$$-P^2 + 100P - 200 = 0$$

$$P^2 - 100P + 200 = 0$$

$$P = \frac{100 \pm \sqrt{100^2 - 4\cdot 200}}{2} = \frac{100 \pm \sqrt{10000 - 800}}{2} = \frac{100 \pm \sqrt{9200}}{2}$$

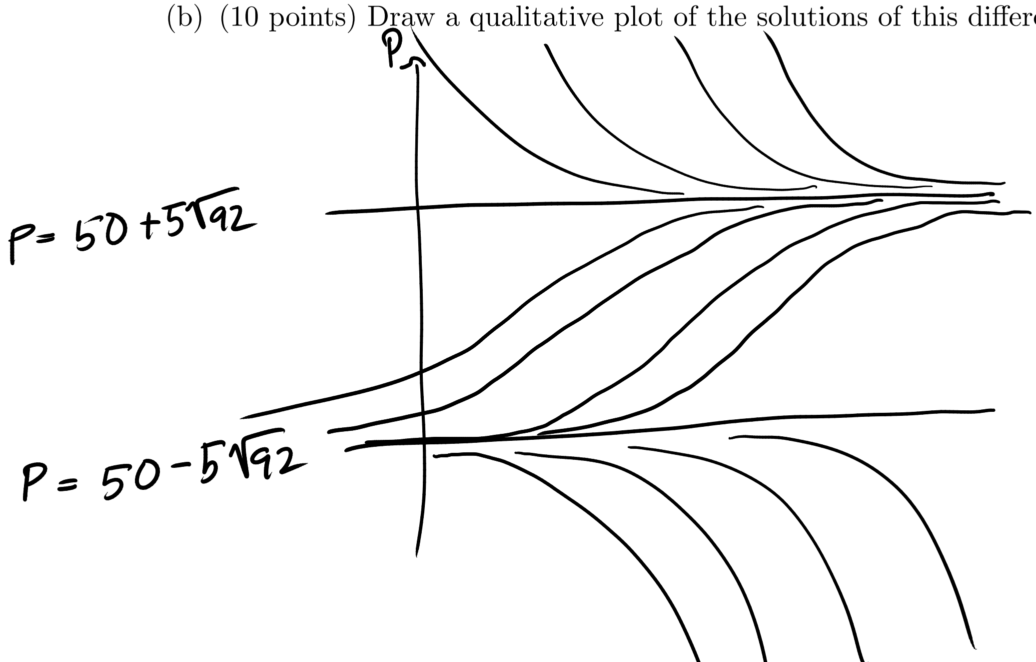
$$= 50 \pm 5\sqrt{92}$$

$$f(P) = -P^2 + 100P - 200$$



$$\boxed{\begin{array}{ll} P = 50+5\sqrt{92} & \text{is } \underline{\text{stable}} \\ P = 50-5\sqrt{92} & \text{is } \underline{\text{unstable}}. \end{array}}$$

(b) (10 points) Draw a qualitative plot of the solutions of this differential equation.



5. (20 points) A metal ball has been heated to $500°C$. It is placed into a bath of water at $30°C$. After 5 seconds, it has cooled to a temperature of $200°C$.

Suppose now that the metal ball is cooled to $0°C$, and again placed into a bath of water at $30°C$. How long will it take to reach a temperature of $20°C$? Your answer does not need to be simplified.

In both situations, the process is governed by Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - A)$$

where A is the temperature of the water, and k is a constant.

$\frac{dT}{dt} = -k(T-A)$

$\int \frac{dT}{T-A} = \int -k\,dt$

$\ln|T-A| = -kt + C$

$|T-A| = e^{-kt} e^{C}$

$T-A = Ce^{-kt}$

$T = A + Ce^{-kt}$

First case $A=30$

$T(0) = 500$

$T(5) = 200$

$500 = 30 + Ce^{-k0}$

$470 = C$

$200 = 30 + 470e^{-k5}$

$170 = 470e^{-5k}$

$e^{5k} = \frac{470}{170} = \frac{47}{17}$

$\boxed{k = \frac{1}{5}\ln\left(\frac{47}{17}\right)}$

Second case $A = 30$

$T(0) = 0$

$0 = 30 + Ce^{0}$

$-30 = C$

$T(t) = 30 - 30e^{-kt}$

$T(t) = 20$

$20 = 30 - 30e^{-kt}$

$30e^{-kt} = 10$

$e^{-kt} = \frac{1}{3}$

$-kt = \ln\left(\frac{1}{3}\right) = -\ln(3)$

$t = \frac{\ln(3)}{k}$

$$\boxed{t = \frac{\ln(3)}{\frac{1}{5}\ln\left(\frac{47}{17}\right)}}$$

This page is for work that doesn't fit on the other pages.