Lecture 9:

Connections III:
$$\begin{tikzcd}
\text{Covariant derivative} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Horizontal distribution} \arrow[d, Rightarrow] \\
& \text{Parallel transport.}
\end{tikzcd}$$

Let $\pi : E \to M$ be a <u>vector bundle</u> over M. Forgetting the vector space structures, it becomes a smooth fiber bundle with typical fiber $\mathbb{R}^q$.

Let $\nabla : \mathfrak{X}(M) \times \Gamma(M,E) \longrightarrow \Gamma(M,E)$ be a covariant derivative. We now claim that ∇ gives rise to a horizontal distribution in E, hence a collection of parallel transport maps. We also wish to express the parallel transport directly in terms of ∇.

<u>Def</u>: a section $s \in \Gamma(M,E)$ is <u>horizontal at $p \in M$ in the direction $X \in T_pM$</u> if $(\nabla_X s)(p) = 0$

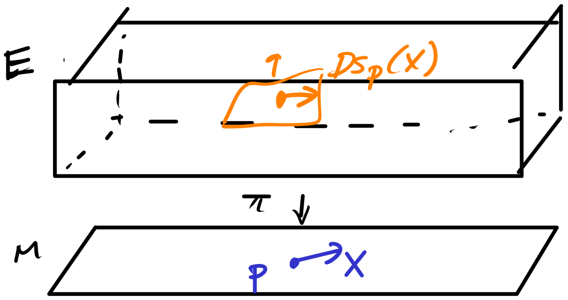
<u>Slogan</u>: horizontality is expressed by vanishing of the covariant derivative.

<u>"Def"</u> The <u>horizontal distribution associated to ∇, H_∇</u> is defined for $q \in E$

$$(H_\nabla)_q := \left\{ Ds_p(X) \;\middle|\; \begin{array}{l} p = \pi(q) \\ s \in \Gamma(M,E) \text{ s.t. } (\nabla_X s)(p) = 0 \end{array} \right\}$$

where $Ds_p : T_pM \to T_qE$ is the differential of s as a smooth map $s : M \to E$.

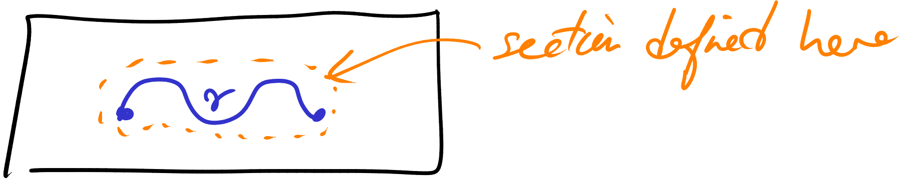
Picture



The word "Def" is in quotation marks because it is not really clear that this defines a subbundle $H_\nabla \subset TE$. Below we show that it is.

By definition, parallel transport for ∇ is the parallel transport for the horizontal distribution H_∇.

There is a direct interpretation of parallel transport in terms of ∇. Let $\gamma : [a,b] \longrightarrow M$ be a path, and suppose that $\tilde{\gamma} : [a,b] \to E$ is a horizontal lift. We may extend $\tilde{\gamma}$ to a section s of E defied in a neighborhood of $\tilde{\gamma}$: $s(\gamma(t)) = \tilde{\gamma}(t)$.



The condition that $\frac{d\tilde{\gamma}}{dt}(t) \in (H_\nabla)_{\tilde{\gamma}(t)}$ and $\frac{d\tilde{\gamma}}{dt}$ is a lift of $\frac{d\gamma}{dt}$ is equivalent to $Ds_{\gamma(t)}\left(\frac{d\gamma}{dt}(t)\right) \in (H_\nabla)_{\tilde{\gamma}(t)}$

By definition of H_∇, this is equivalent to

$$\left(\nabla_{\left(\frac{d\gamma}{dt}(t)\right)} s\right)(\gamma(t)) = 0$$

<u>In summary</u>: To find a horizontal lift $\tilde{\gamma}$ of γ, it is equivalent to find a section s defined in a neighborhood of γ, such that, at any point of γ, the covariant derivative of s in the direction of the velocity vector $\frac{d\gamma}{dt}$ is zero. Then $\underline{\tilde{\gamma}(t) = s(\gamma(t))}$.

Below is one way of checking $H\nabla$ is a horizontal destribution.

To see that it does, we use the local desription of the covariant derivative in terms of coords $p = (x^1, \dots, x^n)$ in $U \subset M$ and a local frame $s_1, \dots, s_r$ of E:
let $e_i = \frac{\partial}{\partial x^i}$. Define $\Gamma_{ij}^k(p)$ by the relation for $p \in U \subset M$

$$\left(\nabla_{e_i} s_j\right)(p) = \sum_{k=1}^{r} \Gamma_{ij}^{k}(p)\, s_j(p)$$

Then for $X(p) = \sum_{i=1}^{n} x^i(p) e_i$ and $s(p) = \sum_{j=1}^{r} a^j(p)\, s_j(p)$

$$\nabla_X s = \nabla_{\sum_i x^i e_i}\Big(\sum_j a^j s_j\Big) = \sum_i x^i\, \nabla_{e_i}\Big(\sum_j a^j s_j\Big)$$

$$= \sum_i x^i \left(\sum_j \left(\frac{\partial a^j}{\partial x^i} s_j + a^j \nabla_{e_i} s_j\right)\right)$$

$$= \sum_i x^i \left(\sum_j \frac{\partial a^j}{\partial x^i} s_j + \sum_j \sum_k a^j \Gamma_{ij}^k s_k\right)$$

rename index and switch

$$= \sum_i x^i \left(\sum_k \frac{\partial a^k}{\partial x^i} s_k + \sum_k \sum_j a^j \Gamma_{ij}^k s_k\right)$$

$$= \sum_{k=1}^{r} \sum_{i=1}^{n} X^i \left(\frac{\partial a^k}{\partial x^i} + \sum_{j=1}^{r} a^j \Gamma_{ij}^{k} \right) s_k$$

Thus $(\nabla_X s)(p) = 0$ iff

$$\forall k=1,\dots,r, \quad \sum_{i=1}^{n} X^i(p) \left(\frac{\partial a^k}{\partial x^i}(p) + \sum_{j=1}^{r} a^j(p)\Gamma_{ij}^{k}(p) \right) = 0.$$

On the other hand, what is $Ds_p(X)$? In the local trivialization

$$\pi^{-1}(U) \simeq U \times \mathbb{R}^r$$

$$\sum_{j=1}^{r} y^j s_j(x^1,\dots,x^n) \mapsfrom (x^1,\dots,x^n, y^1,\dots,y^r)$$

The map $s : U \to \pi^{-1}(U)$ has the form

$$s(x^1,\dots,x^n) = (x^1,\dots,x^n, a^1(x^1,\dots,x^n),\dots, a^r(x^1,\dots,x^n))$$

So $$Ds_p\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \sum_{k=1}^{r} \frac{\partial a^k}{\partial x^i} \frac{\partial}{\partial y^k}$$

and $$Ds_p(X) = Ds_p\left(\sum_i X^i \frac{\partial}{\partial x^i}\right)$$

$$= \sum_i X^i \frac{\partial}{\partial x^i} + \sum_i \sum_k X^i \frac{\partial a^k}{\partial x^i} \frac{\partial}{\partial y^k}$$

$$= X(p) + \sum_k \left(\sum_i X^i(p) \frac{\partial a^k}{\partial x^i}(p) \right) \frac{\partial}{\partial y^k}$$

With these formulas in hand, suppose $(\nabla_X s)(p) = 0$ (that is, s is horizontal in the direction $X(p) \in T_pM$.)

This is equivalent to

$$\forall k : \sum_{i=1}^{n} X^i(p)\left(\frac{\partial a^k}{\partial x^i}(p) + \sum_{j=1}^{r} a^j(p)\,\Gamma_{ij}^{k}(p)\right) = 0.$$

iff

$$\forall k \quad \sum_i X^i(p)\,\frac{\partial a^k}{\partial x^i}(p) = -\sum_i X^i(p)\sum_j a^j(p)\,\Gamma_{ij}^{k}(p)$$

This condition implies

$$Ds_p(X) = X + \sum_k\left(\sum_i X^i(p)\,\frac{\partial a^k}{\partial x^i}(p)\right)\frac{\partial}{\partial y^k}$$

$$= X + \sum_k\left(-\sum_i X^i(p)\sum_j a^j(p)\,\Gamma_{ij}^{k}(p)\right)\frac{\partial}{\partial y^k}$$

Now use that $a^j(p)$ is the value of the coordinate y^j at $s(p) \in E$. So we can also write

$$Ds_p(X) = X - \sum_k\left(\sum_i \sum_j X^i y^j\,\Gamma_{ij}^{k}(p)\right)\frac{\partial}{\partial y^k}$$

This allows us to describe H_∇ without reference to sections. H_∇ consists, at $(x^1,\dots,x^n,y^1,\dots,y^n) \in E$, of all vectors of the form

$$X - \sum_k\left(\sum_i \sum_j X^i y^j\,\Gamma_{ij}^{k}(p)\right)\frac{\partial}{\partial y^k}$$

when $X = X^i e_i$ ranges over T_pM.

Since Γ_{ij}^{k} is a smooth function of $p = (x^1,\dots,x^n)$, H_∇ is a smooth subbundle.