Lecture 8

Interlude: Tensor linear algebra and tensors on manifolds.

We have already used some aspects of tensor algebra, but it seems that not everyone is familiar with it.

* Thoughout this lecture, all vector spaces are finite dimensional *
* over $\mathbb{R}$. *

Let V and W be vector spaces. The <u>tensor product</u> $V \otimes W$ is the vector space generated by symbols $v \otimes w$ for all $v \in V$ and $w \in W$, subject to the relations

(1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

(2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

(3) $\forall r \in \mathbb{R} \quad (rv) \otimes w = r(v \otimes w) = v \otimes (rw)$

<u>Prop</u>: If $e_1, \dots, e_n$ is a basis for V and $f_1, \dots, f_m$ is a basis for W, then the set

$$\left\{ e_i \otimes f_j \;\middle|\; \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, m \end{matrix} \right\}$$

is a basis of $V \otimes W$.

<u>Cor</u> $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Let V, W and Z be vector spaces. A map $A : V \times W \to Z$ is called <u>bilinear</u> if it is liner in each factor separately. that is, for fixed $w \in W$ $A(-, w) : V \to Z$ is liner:

$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$
$$A(rv, w) = r A(v, w)$$

and for fixed $v \in V$, $A(v, -) : W \to Z$ is liner.

<u>Note</u> a bilinear map is <u>not linear</u>: $A(v_1 + v_2, w_1 + w_2) \neq A(v_1, w_1) + A(v_2, w_2)$
rather it equals $A(v_1, w_1) + A(v_2, w_1) + A(v_1, w_2) + A(v_2, w_2)$

The <u>universal property of the tensor product</u> may be formulated by saying that bilinear maps $V\times W\to Z$ are the "same" as linear maps $V\otimes W\to Z$: There is a natural bilinear map

$$V\times W \to V\otimes W \qquad (v,w)\longmapsto v\otimes w$$

For any Z and any multilinear map $A: V\times W\to Z$, there is a unique linear map $A_\otimes: V\otimes W\to Z$ making the diagram commute:

$$\begin{tikzcd} V\times W \arrow[r, "A"] \arrow[d] & Z \\ V\otimes W \arrow[ur, "\exists!\, A_\otimes"'] & \end{tikzcd}$$

The tensor product operation is associative in the sense that

$U\otimes(V\otimes W)$ is canonically isomorphic to $(U\otimes V)\otimes W$

We do not distinguish and write $U\otimes V\otimes W$. (Indeed, all these objects satisfy a universal property with respect to multilinear maps $U\times V\times W\to Z$.)

There are also canonical isomorphisms $V\otimes W\cong W\otimes V$.
$\qquad v\otimes w\mapsto w\otimes v$

and $V\otimes \mathbb{R}\cong V$
$\qquad v\otimes r\to rv$

The <u>dual</u> of V is $V^\vee = \mathrm{Hom}(V,\mathbb{R})$. It is also denoted V^* or V'.

<u>Prop</u> $V\cong (V^\vee)^\vee$ (where as always V is finite dimensional)

The map is $v\in V\longmapsto \underbrace{\left(\lambda\in V^\vee\mapsto \lambda(v)\in\mathbb{R}\right)}_{\in (V^\vee)^\vee}$

Suppose that $B: V\times W\to\mathbb{R}$ is a bilinear map. (Or equivalently $B_\otimes: V\otimes W\to\mathbb{R}$ is a linear map). Then we obtain a map

$\tilde{B}: W\to V^\vee \qquad \tilde{B}(w) = \left(v\mapsto B(v,w)\right)\in V^\vee$

$\left[\text{or} \quad \tilde{B}(w) = \left(v\longmapsto B_\otimes(v\otimes w)\right)\right]$

The bilinear map $B: V \times W \to \mathbb{R}$ is called a <u>perfect pairing</u> if the associated map $\tilde{B}: W \to V^\vee$ is an isomorphism (equivalently the map $V \to W^\vee$ defined similarly is an isomorphism)

<u>Observe</u>: The existence of a perfect pairing implies V and W have the same dimension.

There is an obvious perfect pairing $ev: V^\vee \otimes V \to \mathbb{R}$ called <u>evaluation</u> or <u>contraction</u>.
it is defined by $ev(\alpha \otimes \beta) = \alpha(\beta)$ where $\alpha \in V^\vee, \beta \in V$

<u>Prop</u> Let $g: V \times V \longrightarrow \mathbb{R}$ be a nondegenerate bilinear form then g is a perfect pairing:

<u>Pf</u> nondegeneracy means that $\forall v \in V \; \exists w \in V$ such that $g(v, w) \neq 0$. This exactly says that the map
$$\tilde{g}: V \to V^\vee \quad \tilde{g}(v) = (w \longmapsto g(v, w))$$
has kernel $= \{0\}$. Since $\tilde{g}$ is an injective map between spaces of the same dimension, it is an isomorphism. ▨

Along similar lines we have the following very useful proposition
<u>Prop</u> There is a canonical isomorphism $V^\vee \otimes W \xrightarrow{\sim} Hom(V, W)$.
<u>Construction</u>: There is an obvious bilinear map
$$V^\vee \times W \to Hom(V, W)$$
$$(\lambda, w) \mapsto (v \mapsto \lambda(v)\, w)$$
By universal property, we get a map $V^\vee \otimes W \to Hom(V, W)$

<u>Proof</u> that this is an isomorphism: First note that $V^\vee \otimes W$ and $\mathrm{Hom}(V,W)$ are vector spaces of the same dimension (namely, $\dim(V)\cdot\dim(W)$). So it will suffice to prove that the map is surjective.

For that, let $e_1,\ldots,e_n$ be a basis of V.

Any $v \in V$ can be written uniquely as $v = a_1e_1 + \cdots + a_ne_n$ for some coefficients $a_i \in \mathbb{R}$

There is a basis of $V^\vee$ called $e_1^\vee,\ldots,e_n^\vee$, where

$$e_i^\vee(v) := \begin{pmatrix} \text{the coefficient of } e_i \text{ when } v \\ \text{is represented in the basis } e_1,\ldots,e_n \end{pmatrix} = a_i$$

In short, we have an identity $v = e_1^\vee(v)e_1 + \cdots + e_n^\vee(v)e_n$

Now let $\varphi \in \mathrm{Hom}(V,W)$. Then $\varphi(e_1),\ldots,\varphi(e_n) \in W$.

We find
$$\begin{aligned}\varphi(v) &= \varphi\left(e_1^\vee(v)e_1 + \cdots + e_n^\vee(v)e_n\right)\\ &= e_1^\vee(v)\varphi(e_1) + \cdots + e_n^\vee(v)\varphi(e_n)\end{aligned}$$

Then $e_1^\vee \otimes \varphi(e_1) + \cdots + e_n^\vee \otimes \varphi(e_n) \in V^\vee \otimes W$ maps to φ under the map $V^\vee \otimes W \to \mathrm{Hom}(V,W)$ constructed above. ▨

We also have a canonical isomorphism
$$\mathrm{Hom}(U, \mathrm{Hom}(V,W)) \cong \mathrm{Hom}(U \otimes V, W)$$
(This says that $\mathrm{Hom}(V,-)$ and $-\otimes V$ are adjoint functors.)

<u>Exercise</u>: $(V \otimes W)^\vee \cong V^\vee \otimes W^\vee$

Now look what we can do: Let U, V, W, X, Y, Z be vector spaces

$$U \otimes V^\vee \otimes W^\vee \otimes X \otimes Y \otimes Z^\vee \cong V^\vee \otimes U \otimes W^\vee \otimes X \otimes Y \otimes Z^\vee$$
$$\cong \mathrm{Hom}(V, U \otimes W^\vee \otimes X \otimes Y \otimes Z^\vee) \cong \mathrm{Hom}(V, W^\vee \otimes Z^\vee \otimes U \otimes X \otimes Y)$$
$$\cong \mathrm{Hom}(V, \mathrm{Hom}(W, Z^\vee \otimes U \otimes X \otimes Y))$$
$$\cong \mathrm{Hom}(V, \mathrm{Hom}(W, \mathrm{Hom}(Z, U \otimes X \otimes Y)))$$
$$\cong \mathrm{Hom}(V, \mathrm{Hom}(W \otimes Z, U \otimes X \otimes Y))$$
$$\cong \mathrm{Hom}(V \otimes W \otimes Z, U \otimes X \otimes Y)$$

It's also isomorphic to $\mathrm{Hom}(U^\vee \otimes V \otimes W \otimes X^\vee \otimes Y^\vee, Z^\vee)$, say.

Under the isomorphism $V^\vee \otimes V \xrightarrow{\sim} \mathrm{Hom}(V,V)$, the evaluation $V^\vee \otimes V \to \mathbb{R}$ corresponds to the <u>trace</u>
Pick $e_1, \dots, e_n$ a basis of V, $e_1^\vee, \dots, e_n^\vee$ the dual basis

Then for $\varphi \in \mathrm{Hom}(V,V)$; $tr(\varphi) = \sum_{i=1}^n e_i^\vee(\varphi(e_i))$
(This does not depend on the choice of basis.)

More <u>general contraction</u>: Consider $V \otimes W \otimes X \otimes W^\vee \otimes Y$
Ah! we see a dual pair W and $W^\vee$. We can contract them using the evaluation $W^\vee \otimes W \to \mathbb{R}$, and we get a map

$$V \otimes W \otimes X \otimes W^\vee \otimes Y \to V \otimes X \otimes Y \otimes \mathbb{R} \simeq V \otimes X \otimes Y$$

<u>Choices</u>. Consider $V \otimes V^\vee \otimes V^\vee$ Can contract first and second factors <u>or</u> first and third factors $\to$ Get different maps $V \otimes V^\vee \otimes V^\vee \to V^\vee$

Tensor algebra of a single vector space V

$$V^{\otimes k} := \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \qquad V^{\otimes 0} := \mathbb{R}$$
$$V^{\otimes 1} := V$$

The <u>Tensor algebra</u> $T(V) := \bigoplus_{k=1}^{\infty} V^{\otimes k}$

it is a noncommutative <u>ring</u>:

$$\underbrace{(v_1 \otimes \cdots \otimes v_k)}_{\in V^{\otimes k}} \cdot \underbrace{(v_{k+1} \otimes \cdots \otimes v_{k+l})}_{\in V^{\otimes l}} = \underbrace{v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}}_{\in V^{\otimes (k+l)}}$$

(more precisely, $T(V)$ is a $\mathbb{Z}$-graded $\mathbb{R}$-algebra)

With respect to a basis $e_1, \ldots, e_n$ of V, an element $t \in V^{\otimes k}$ looks like

$$t = \sum_{i_k=1}^{n} \cdots \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} t^{i_1 i_2 \cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$$

We also have the <u>bivariant tensor algebra</u>

$$\tilde{T}(V) = \bigoplus_{k=1}^{\infty} \bigoplus_{l=1}^{\infty} V^{\otimes k} \otimes (V^\vee)^{\otimes l} \qquad (\mathbb{Z}^2\text{-graded})$$

With respect to a basis $e_1, \ldots, e_n$ and dual basis $e_1^\vee, \ldots, e_n^\vee$ an element $t \in V^{\otimes k} \otimes (V^\vee)^{\otimes l}$ looks like

$$t = \sum_{j_1 \cdots j_l = 1}^{n} \sum_{i_1 \cdots i_k = 1}^{n} t^{i_1 \cdots i_k}_{j_1 \cdots j_l} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^\vee_{j_1} \otimes \cdots \otimes e^\vee_{j_l}$$

Over a manifold M, we replace vector spaces by vector bundles. it's the same, but everything now depends also on the point $x \in M$.

When $V \rightsquigarrow TM$ then sections of $TM^{\otimes k}$ are called <u>contravariant</u> k-tensors on M, while sections of $T^*M^{\otimes k}$ are called <u>covariant</u> k-tensors on M. Sections of $(TM)^{\otimes k} \otimes (T^*M)^{\otimes \ell}$ are said to have <u>mixed variance</u>.