Lecture 7:

Connections II: Horizontal Distributions & Parallel transport

If covariant derivatives weren't geometric enough for you, you're in luck. Another approach to connections is through <u>Horizonal distributions</u>. It actually applies to more general fiber (not necessarily vector!) bundles

<u>Def</u> Let M be a smooth manifold. Let F be a smooth manifold. A <u>smooth fiber bundle over M with typical fiber F</u> consist of

- E a smooth manifold
- $\pi : E \to M$ a smooth map.

Such that π is locally trivial:

M is covered by open sets U_α such that for each α, we have a map $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that

(1) $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism

(2) The diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times F \arrow[d, "pr_1"] \\ U_\alpha \arrow[r, equal] & U_\alpha \end{tikzcd}$$

<u>Remarks</u>:
- Consequences of the local triviality condition are that π is a surjective submersion, and that, for every $x \in M$, $\pi^{-1}(x)$ is diffeomorphic to F, that is, <u>all fibers are diffeomorphic to F</u>.
- A vector bundle, forgetting the vector space structure on the fibers, yields a smooth fiber bundle with typical fiber $\mathbb{R}^k$. (But a vector bundle is more than that)

Let $\pi: E \to M$ be a smooth fiber bundle with typical fiber F. Inside TE = tangent bundle of E, there is a subbundle $(TE)^v$ of <u>vertical vectors</u>, that is vectors tangent to the fibers of π, that is vectors in the kernel of $D\pi$

$$(TE)^v = \ker(D\pi), \qquad (TE)^v_x = T_x(\pi^{-1}(x))$$

Observe:
- $\operatorname{rank}(TE)^v = \dim F$.
- There is an exact sequence of vector bundles on E:

$$0 \to (TE)^v \to TE \xrightarrow{D\pi} \pi^* TM \to 0$$

<u>Def</u>: A <u>horizontal distribution</u> (or <u>connection</u>) on E is a subbundle $H \subset TE$ that is complementary to $(TE)^v$: That is

$$\forall x \in E, \quad H_x \text{ and } (TE)^v_x \text{ span } TE_x$$
$$\text{and} \quad H_x \cap (TE)^v_x = \{0\}.$$
$$\text{so} \quad TE_x \simeq H_x \oplus (TE)^v_x$$

We can also think of this as a Whitney sum decomposition:
$$TE \simeq (TE)^v \oplus H.$$
Or as a splitting of the exact sequence
$$0 \to (TE)^v \to TE \to \pi^* TM \to 0$$
Recall that a splitting of an exact sequence
$$0 \to V' \xrightarrow{i} V \xrightarrow{p} V'' \to 0$$
is a right inverse of p
$$\begin{tikzcd} V \arrow[r, "p"] & V'' \arrow[l, bend left, "s"] \end{tikzcd} \qquad p \circ s = \mathrm{id}_{V''}$$

or a left inverse of i
$$\begin{tikzcd} V' \arrow[r, "i"] & V \arrow[l, bend left, "r"] \end{tikzcd} \qquad r \circ i = \mathrm{id}_{V'}.$$

When a horizontal distribution H is given, a left inverse to

$(TE)^v \xrightarrow{i} TE$ is given by $\omega : TE \to (TE)^v$

$\omega(X) = X^v$, where $X = X^v + X^h$ is the unique decomposition of X into a vertical vector $X^v \in (TE)^v$ and a horizontal vector $X^h \in H$

That is we project to $(TE)^v$ <u>along</u> H.

$$\left.\begin{array}{l} X^v = \text{"vertical component of } X\text{"} \\ X^h = \text{"horizontal component of } X\text{"} \end{array}\right\} \text{both depend on choice of connection.}$$

This ω may also be thought of as a section of

$$\mathcal{H}om(TE, (TE)^v) = T^*E \otimes (TE)^v$$

$$\omega \in \Gamma(E, T^*E \otimes (TE)^v) = \Omega^1(E, (TE)^v)$$

"ω is a 1-form on E with values in vertical vector fields."

It satisfies $\omega(X) = 0 \iff X \in H$ and $\omega(X) = X \iff X \in (TE)^v$

A right inverse to $TE \xrightarrow{D\pi} \pi^*TM$ is obtained by noting that $(D\pi)_x$ maps H_x isomorphically onto $T_{\pi(x)}M$.

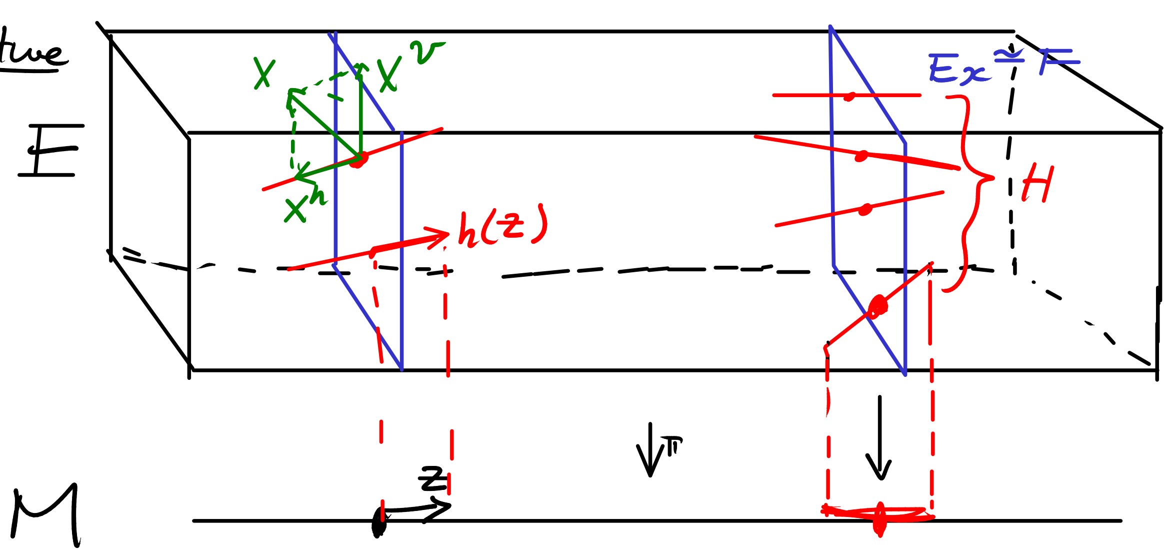
Inverting this isomorphism gives

$$h : \pi^*TM \longrightarrow H \subset TE$$

for $Z \in (\pi^*TM)_x$, $h(Z) =$ the unique vector in H_x such that $D\pi(h(Z)) = Z$

$h(Z)$ is called the <u>horizontal lift</u> of Z.

<u>Picture</u>



The concept of horizontal lifting leads to the notion of <u>parallel transport</u>.
Let $\pi : E \to M$ be a smooth fiber bundle with connection H.
Let $\gamma(t) : t \in [a,b] \to M$ be a smooth path in M.

<u>Def</u> A <u>horizontal lift</u> of γ is a path $\tilde{\gamma} : [a,b] \to E$ such that

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

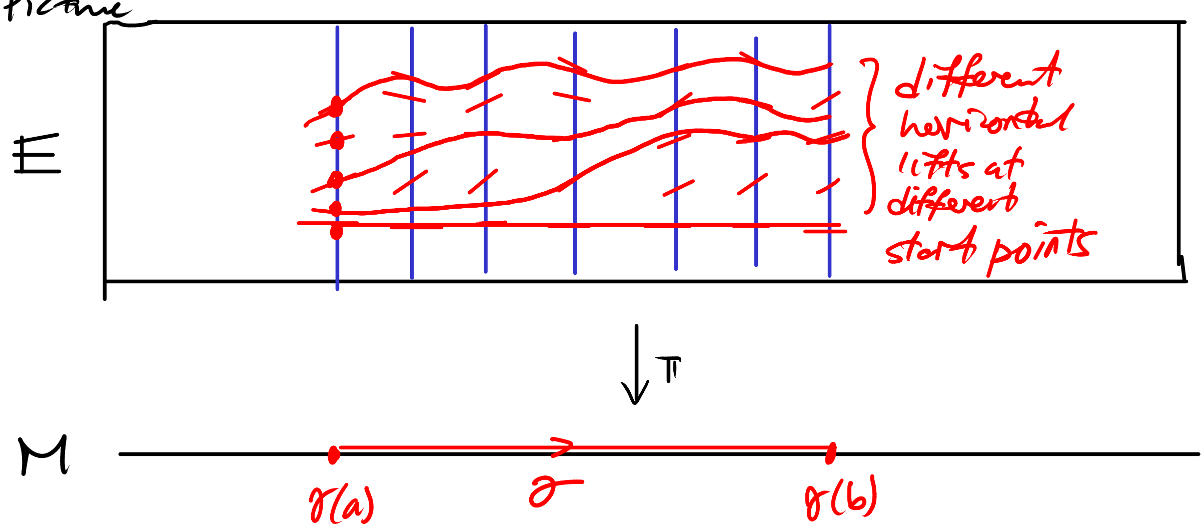
commutes (i.e. $\pi \circ \tilde{\gamma} = \gamma$)

and such that the tangent vector of $\tilde{\gamma}$ is horizontal:

$$\frac{d\tilde{\gamma}}{dt}(t) \in H_{\tilde{\gamma}(t)} \quad \text{for all } t \in [a,b]$$

(it follows that $\frac{d\tilde{\gamma}}{dt}(t)$ is indeed $h\left(\frac{d\gamma}{dt}(t)\right)$, the horizontal lift of the tangent vector to γ at the point $\tilde{\gamma}(t)$).

Picture



Analytically, the condition that $\tilde{\gamma}$ be a horizontal lift of γ is a first order non-linear ordinary differential eqn.

$$\frac{d\tilde{\gamma}}{dt} = h\left(\frac{d\gamma}{dt}\right)_{\tilde{\gamma}(t)} \quad (\text{horizontal lift at } \tilde{\gamma}(t))$$

Locally it's like $\frac{d\vec{x}}{dt} = f(t, \vec{x})$ studied in math 285.

Thus we expect the initial value problem

$$(\mathrm{HL}) \quad \begin{cases} \tilde{\gamma} \text{ is a horizontal lift of } \gamma : [a,b] \to M \\ \tilde{\gamma}(a) = x \in \pi^{-1}(\gamma(a)) \end{cases}$$

To be well-posed for any path γ and starting value $x \in \pi^{-1}(\gamma(a))$

Fact: · Uniqueness holds: the solution of (HL) is unique if it exists.

- Short-time existence holds: $\exists\, \varepsilon > 0$ such that $\tilde{\gamma}(t)$ solving (HL) exists for $t \in [a, a+\varepsilon)$
- long-time existence, existence of $\tilde{\gamma}$ for all $t \in [a,b]$, does not hold in this level of generality, but it does hold in all cases we will consider. In particular, it holds if the fiber F is compact, or if E is a vector bundle and the connection "comes from a covariant derivative."

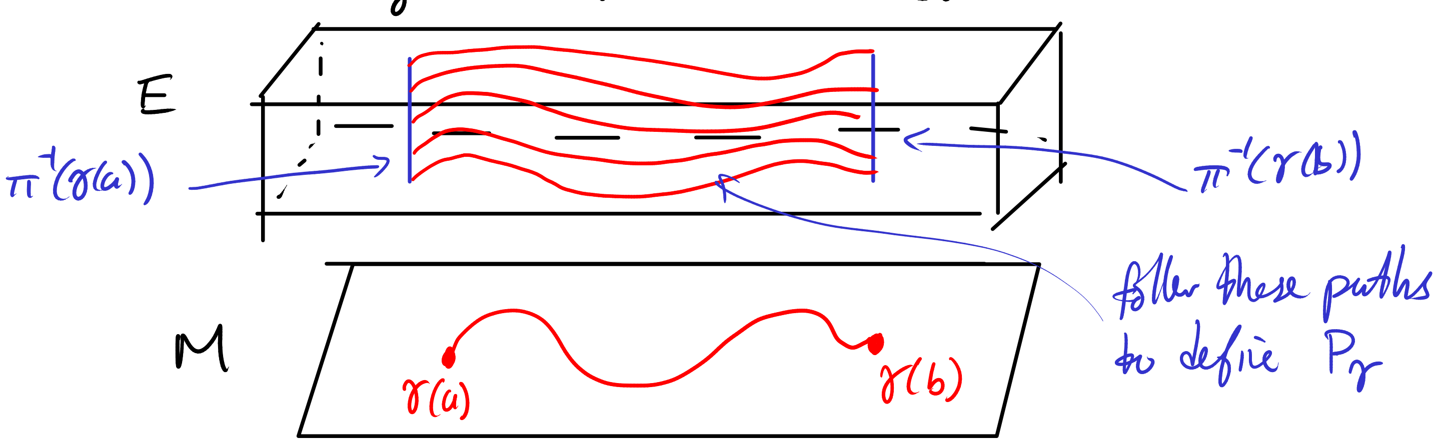
Def Suppose that $\tilde{\gamma} : [a,b] \to M$ is a horizontal lift of γ satisfying $\tilde{\gamma}(a) = x \in \pi^{-1}(\gamma(a))$. Then $y = \gamma(b) \in \pi^{-1}(\gamma(b))$ is called the parallel transport of x along γ

We write

$$y = P_\gamma(x).$$

Provided horizontal lifts exist for all time, this correspondence defines a map

$$P_\gamma : \pi^{-1}(\gamma(a)) \longrightarrow \pi^{-1}(\gamma(b))$$



Prop: the map $P_\gamma : \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(b))$ is a diffeomorphism

Proof: By uniqueness of horizontal lifts, P_γ is well-defined.
We can reverse the parametrization of the path:
Define $\bar{\gamma} : [-b,-a] \to M$ to be $\bar{\gamma}(t) = \gamma(-t)$
Then $\bar{\gamma}$ is a path from $\gamma(b)$ to $\gamma(a)$
If $\tilde{\gamma}$ is a horizontal lift of γ, then $\overline{(\tilde{\gamma})}$ is a horizontal lift of $\bar{\gamma}$. Thus

$$y = P_\gamma(x) \iff x = P_{\bar{\gamma}}(y)$$

This shows that P_γ is bijective. To see that P_γ is smooth, we must appeal to the fact that the solution of a first order ODE $\left\{ \begin{array}{l} \frac{d\vec{x}}{dt} = f(t,\vec{x}) \\ \vec{x}(0) = \vec{x}_0 \end{array} \right\}$ depends smoothly on the initial value, provided the right hand side $f(t,\vec{x})$ is smooth. (Analyse the Picard iteration.) ∎

Thus, parallel transport lets us identify different fibers of $\pi : E \to M$, but the identification depends on the path γ.

Obvious generalization: allow γ to be piecewise smooth

$$P_\gamma = P_{\gamma_4} \circ P_{\gamma_3} \circ P_{\gamma_2} \circ P_{\gamma_1}$$

Holonomy: let γ be a loop in M based at $x \in M$:
$\gamma : [a,b] \to M \quad \gamma(a) = \gamma(b) = x$, write $E_x = \pi^{-1}(x)$
Then $P_\gamma : E_x \to E_x$ is a diffeomorphism of E_x.
Considering all piecewise smooth loops based at x, we get the Holonomy group at x of the connection H.

$$\mathrm{Hol}_x(H) = \left\{ P_\gamma \;\middle|\; \gamma \text{ is p.w. smooth loop based at } x \right\} \subseteq \mathrm{Diff}(E_x)$$

Note that $\mathrm{Hol}_x(H)$ is a group, since composition of loops corresponds to composition of parallel transport maps, and reversal of loops corresponds to inverses.

An important idea is that connections with "small" holonomy group correspond to "structures" on the bundle (vector space structure, orthogonal structure, flat structure, ...)