Lecture 6:

Connections I: Covariant Derivatives

We now come to connections, one of the crucial notions in differential geometry. There are roughly 4 different and closely related versions of a connection:

1) Covariant derivative (Linear connection in a vector bundle)
2) Principal connection in a Principal G-bundle.
3) Horizontal distribution (Ehresmann connection)
4) Parallel Transport Maps.

[N.B. They are not all exactly equivalent]

Covariant derivatives. As motivation, consider

M = smooth manifold

$C^\infty(M)$ = smooth real valued functions on M.

$\mathfrak{X}(M)$ = smooth vector fields on M.

Algebra:

$C^\infty(M)$ is a commutative ring, and a module for $\mathfrak{X}(M)$

Action $\mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M)$

$(X, f) \longmapsto X.f$

where $X.f$ denotes directional derivative of f in direction X.

in local coordinates $(x^1, \dots, x^n)$

$$X = \sum_{i=1}^{n} X^i(x) e_i \qquad \left(e_i = \frac{\partial}{\partial x^i} \text{ coordinate frame} \right)$$

$$X.f = \sum_{i=1}^{n} X^i(x) \frac{\partial f}{\partial x^i}$$

This action has the algebraic property of being a derivation

$$X.(fg) = (X.f)g + f(X.g)$$

"product rule"
"Leibniz rule"

(For R a ring, have derivations
$$\mathrm{Der}(R) = \{ P : R \to R \mid P \text{ linear}, \quad P(ab) = P(a)b + aP(b) \})$$

Thus there is a map $\quad \mathfrak{X}(M) \longrightarrow \mathrm{Der}(C^\infty(M))$
It is in fact a homomorphism of Lie algebras.
(w/ right sign convention)

Thus vector fields yield "first-order differential operators" on $C^\infty(M)$
We want to extend this to vector bundles, where $C^\infty(M)$ is replaced by the space of smooth sections.

$\pi : E \to M$ = smooth vector bundle
$\Gamma(M, E)$ = smooth sections $s : M \to E$.

By fiber wise addition an scalar mult, $\Gamma(M,E)$ is a $\mathbb{R}$-vector space
It is also a $C^\infty(M)$-module

$$C^\infty(M) \times \Gamma(M,E) \longrightarrow \Gamma(M,E)$$
$$(f(x), s(x)) \longrightarrow \underbrace{f(x) \cdot s(x)}_{\text{scaler mult in } E_x}$$

Def
A covariant derivative in E is a map.
$$\nabla : \mathfrak{X}(M) \times \Gamma(M,E) \longrightarrow \Gamma(M,E)$$
$$(X, s) \longmapsto \nabla_X s$$
nabla

satisfying: $\mathbb{R}$-linearity $\quad \nabla_X(as_1 + bs_2) = a(\nabla_X s_1) + b(\nabla_X s_2) \quad a,b \in \mathbb{R}$

· Leibniz rule $\quad \nabla_X(f \cdot s) = (X.f)s + f\nabla_X s \quad f \in C^\infty(M)$

· Tensoriality in $\mathfrak{X}(M)$
$$\nabla_{(aX+bY)} s = a\nabla_X s + b\nabla_Y s$$
$$\nabla_{(fX)} s = f\nabla_X s \qquad f \in C^\infty(M)$$

The first two conditions say that ∇_X is a "first-order differential operator" on $\Gamma(M,E)$.

The condition of "Tensoriality" implies that, for a given point $x \in M$, the quantity $(\nabla_X s)(x)$ depends on on $X(x)$, not the values of X at nearby points. (similar to $(X.f)(x)$) (Exercise).

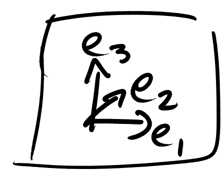
Exercise: Every vector bundle over a (2nd countable Hausdorff) manifold admits a covariant derivative/Connection.

Just as a vector field has a local description $X = \sum_{i=1}^{n} X^i(x) \frac{\partial}{\partial x^i}$ So does a connection.

Let $(x^1, \dots, x^n)$ denote local coordinates on M defined in $U \subset M$

let $(s_1(x), \dots, s_r(x))$ denote a local frame for E, defined in U (i.e. we are assuming E is trivialized over U)

Let $e_i = \frac{\partial}{\partial x^i}$ denote the coordinate vector fields



As $\nabla_{e_i} s_j$ is a section of E, it may be written uniquely as

$$(\nabla_{e_i} s_j)(x) = \sum_{k=1}^{r} \Gamma^k_{ij}(x)\, s_k(x)$$

Prop: the $n \cdot r^2$ functions $\left(\Gamma^k_{ij}(x)\right)_{\substack{i=1,\dots,n \\ j=1,\dots,r \\ k=1,\dots,r}}$ determine ∇ in U.

<u>Proof</u> Consider $X = \sum_{i=1}^{n} X^i(x) e_i \qquad s(x) = \sum_{j=1}^{r} a^j(x) s_j(x)$

Then $\nabla_X s = \nabla_{\sum X^i(x) e_i} s = \sum_{i=1}^{n} X^i(x) \nabla_{e_i} s$

and $\nabla_{e_i} s = \nabla_{e_i}\left(\sum_{j=1}^{r} a^j(x) s_j(x)\right) = \sum_{j=1}^{r} \nabla_{e_i}\left(a^j(x) s_j(x)\right)$

$$= \sum_{j=1}^{r}\left(\frac{\partial a^j}{\partial x^i} s_j(x) + a^j(x) \nabla_{e_i} s_j(x)\right)$$

$$= \sum_{j=1}^{r}\left(\frac{\partial a^j}{\partial x^i} s_j(x) + a^j(x) \sum_{k=1}^{r} \Gamma_{ij}^{k}(x) s_k(x)\right) \qquad \blacksquare$$

The functions $\Gamma_{ij}^k(x)$ are called "coefficients" or "Christoffel symbols" of the connection. They are very much tied to the choice of coordinates and the local frame $\{s_j(x)\}$.
To get something more invariant, we consider the <u>difference</u> of two connections.

<u>Prop</u> Let ∇^1 and ∇^2 be two connections in the same bundle $E \to M$. Then there is a section $a \in \Gamma(M, T^*M \otimes \mathcal{H}om(E,E))$ such that

$$\nabla_X^2 s - \nabla_X^1 s = a(X) \cdot s$$

[<u>Notation</u>: for $X \in \mathfrak{X}(M)$, $a(X)$ denotes pairing the one-form part of a with X. The result is a section of $\mathcal{H}om(E,E)$, which we apply to s]

Pf let $A(X,s) = \nabla^2_X s - \nabla^1_X s$
Since ∇^2 and ∇^1 are connections, we find

$$A(f(x)X + g(x)Y, s) = f(x)A(X,s) + g(x)A(Y,s) \qquad f,g \in C^\infty(M)$$

because both ∇^1 and ∇^2 satisfy

$$\nabla_{f(x)X+g(x)Y} s = f(x)\nabla_X s + g(x)\nabla_Y s$$

Tensoriality in the vector input

But we also have $A(X, fs_1 + gs_2) = fA(X,s_1) + gA(X,s_2)$
for any $f,g \in C^\infty(M)$ (even though ∇^1, ∇^2 don't have this property)

The important thing is to show $A(X,fs) = fA(X,s)$, $f \in C^\infty(M)$
Indeed: $A(X,fs) = \nabla^2_X(fs) - \nabla^1_X(fs)$

$$= X.f + f\nabla^2_X s - (X.f + f\nabla^1_X s)$$

$$= f\nabla^2_X s - f\nabla^1_X s = f(\nabla^2_X s - \nabla^1_X s) = fA(X,s)$$

Thus the function $A(X,s)$ is tensorial in both inputs
This means it necessarily has the form

$$A(X,s) = a(X)\cdot s$$

for some $a \in \Gamma(M, T^*M \otimes \mathcal{H}om(E,E))$. $\blacksquare$

Since we've encountered it a couple of times, here is a prototype for this sort of "tensoriality" argument.

Suppose $\Omega : \Gamma(M,E) \to C^\infty(M)$ is an $\mathbb{R}$-linear map.
Ω is <u>tensorial</u> if F is $C^\infty(M)$-linear:

$$\Omega(fs) = f\Omega(s) \qquad f \in C^\infty(M)$$

<u>Thm</u>: If Ω is tensorial then $\exists!$ $\omega \in \Gamma(M, E^\vee)$
s.t. $\Omega(s)(x) = \langle \omega(x), s(x) \rangle \quad \forall x \in M$
where $\langle \, , \rangle$ denotes fiberwise pairing of $E^\vee_x$ with E_x

(That is, a $C^\infty(M)$-linear functional on sections arises from a section of the bundle of linear functionals)

<u>Proof</u>: <u>Step 1</u> Ω is local: $\operatorname{supp} \Omega(s) \subset \operatorname{supp} s = \overline{\{s \neq 0\}}$
For suppose not: then $\exists\, x_0 \in \operatorname{supp} \Omega(s)$, $x_0 \notin \operatorname{supp} s$
Since $x_0 \notin \operatorname{supp} s$ $\exists$ open $U \ni x_0$ such that $s = 0$ in U
Let φ be a bump function which is 1 near x_0, and 0 outside U. Then $\varphi s = 0$ so $0 = \Omega(\varphi s) = \varphi \Omega(s)$
But $\varphi \Omega(s) = 1 \cdot \Omega(s)$ near x_0 and $x_0 \in \operatorname{supp} \Omega(s)$, so there are points arbitrarily close to x_0 where $\Omega(s) \neq 0$
This is a contradiction.

Thus if U is open and $s(x) = 0 \ \forall x \in U$, $\Omega(s)(x) = 0 \ \forall x \in U$
It follows that if $s_1(x) = s_2(x) \ \forall x \in U$, then $\Omega(s_1)(x) = \Omega(s_2)(x) \ \forall x \in U$

Choosing U small enough, we have a local frame $\{s_j(x)\}_{j=1}^r$.
In U any section s has

$$(s|_U)(x) = \sum_{j=1}^r a^j(x) s_j(x) \quad \text{for some } a^j(x) \in C^\infty(U)$$

$\Omega(s)\big|_U$ is determined by $s|_U$

$$\forall x \in U \quad \Omega(s|_U)(x) = \Omega\Big(\sum_{j=1}^r a^j(x) s_j(x)\Big) = \sum_{j=1}^r a^j(x) \Omega(s_j(x))$$

Now for a single point x_0: $s(x_0) \Rightarrow$ all $a^j(x_0) = 0 \Rightarrow \Omega(s)(x_0) = 0$.

It follows that if $s_1(x_0) = s_2(x_0)$ then $\Omega(s_1)(x_0) = \Omega(s_2)(x_0)$

Thus for each fixed x_0, Ω determines a linear function.

$$\omega(x_0) : E_{x_0} \to \mathbb{R}$$
$$v \to \langle \omega(x_0), v \rangle$$

(choose local section s such that $s(x_0) = v$. then take $\Omega(s)(x) \in \mathbb{R}$ this defines $\langle \omega(x_0), v \rangle$)

We conclude that there is a (possibly discontinuous) section $\omega \in \Gamma(M, E^\vee)$ such that

$$\Omega(s)(x) = \langle \omega(x), s(x) \rangle$$

The fact that ω is actually smooth follows from the fact that $\Omega(s)$ is smooth for every smooth s.