Lecture 5: Metrics

We continue the process of extending notions from vector spaces to vector bundles.

Let V be a vector space over $\mathbb{R}$ of dimension n.

<u>Def</u> A <u>bilinear form</u> on V is a map $b : V \times V \to \mathbb{R}$ which is linear in each factor:

$$\left.\begin{aligned} &\cdot\ b(\lambda u + \mu v, w) = \lambda b(u,w) + \mu b(v,w) \\ &\cdot\ b(w, \lambda u + \mu v) = \lambda b(w,u) + \mu b(w,v) \end{aligned}\right\} \begin{aligned} &\forall\, u,v,w \in V \\ &\forall\, \lambda, \mu \in \mathbb{R} \end{aligned}$$

The bilinear form b is called

- <u>symmetric</u> if $b(u,v) = b(v,u) \quad \forall\, u,v \in V$
- <u>skew-symmetric</u> if $b(u,v) = -b(v,u) \quad \forall\, u,v \in V$
- <u>nondegenerate</u> if $\forall\, u \in V\ \exists\, v \in V$ s.t. $b(u,v) \neq 0$
  (i.e. there is no vector which is b-orthogonal to every vector)
- <u>positive definite</u> if $\forall\, v \in V,\ v \neq 0 \Rightarrow b(v,v) > 0$
  (or: $b(v,v) \geq 0$ and $= 0$ iff $v = 0$)
- a <u>Euclidean metric</u> if b is symmetric & positive definite
- a <u>symplectic form</u> if b is skew-symmetric and nondegenerate.

The set of bilinear forms on V is a vector space of dimension n^2
It may be coordinatized by picking a basis $e_1, \dots, e_n$ of V, and taking the "matrix entries" $b_{ij} = b(e_i, e_j)$
In matrix notation

$$b(u,v) = u^T \cdot [b_{ij}] \cdot v \qquad u, v \in \mathbb{R}^n \text{ (columns)}$$

Thus it makes sense to speak of a "smoothly varying" family of bilinear forms, hence of bilinear forms on a vector bundle

Let $\pi : E \to X$ be a smooth vector bundle over a smooth manifold.

<u>Def</u> A smooth bilinear form on $\pi : E \to X$ is a collection $\{b_x\}_{x \in X}$ such that $b_x : E_x \times E_x \to \mathbb{R}$ is a bilinear form on the fiber $E_x = \pi^{-1}(x)$, and which is smoothly varying in the following sense: let $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a smooth local trivialization; let $\{e_i\}_{i=1}^n$ denote standard basis of $\mathbb{R}^n$. Then we require

$$b_{ij}(x) := b_x\left(\varphi^{-1}(x, e_i), \varphi^{-1}(x, e_j)\right)$$

is a smooth function on U.

(This is required to hold for every local trivialization).

Alternatively, we could use the multilinear algebra for vector bundles developed in the previous lecture. Bilinear forms on V are identified with elements of $V^\vee \otimes V^\vee$. Thus smooth bilinear forms on $E \to X$ are nothing but smooth sections of the vector bundle $E^\vee \otimes E^\vee \to X$.

A smooth bilinear form b on $E \to X$ is called <u>symmetric</u> if each $b_x : E_x \times E_x \to \mathbb{R}$ is symmetric, etc.

Of all the sorts of bilinear forms, the most important are Euclidean metrics. In fact they always exist on vector bundles over (2nd countable hausdorff) manifolds.

Theorem: let $\pi : E \to X$ be a vector bundle over a (2nd countable Hausdorff) Manifold X.
Then E may be given a smooth Euclidean metric.

The idea of proof is to use a partition of unity to glue together standard metrics from the local trivializations. The reason why this works for metrics and not other sorts of forms is the following:

Lemma The set of all Euclidean metrics on a vector space V is convex:
if g_1 and g_2 are metrics, then so is $tg_1 + (1-t)g_2$ for all $0 \le t \le 1$

Pf: Obviously $tg_1 + (1-t)g_2$ is a symmetric bilinear form, since symmetric biliner forms comprise a vector space.
The key is positive definiteness, which holds because

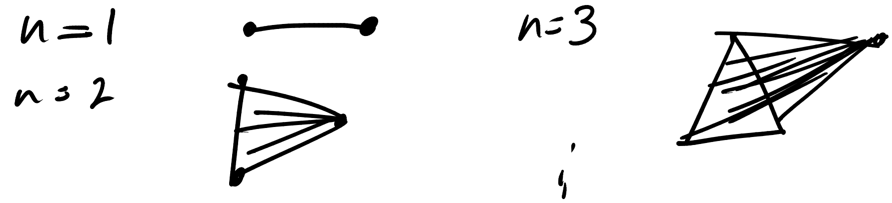
$$\underbrace{t}_{\geq 0}\, \underbrace{g_1(v,v)}_{\geq 0} + \underbrace{(1-t)}_{\geq 0}\, \underbrace{g_2(v,v)}_{\geq 0} \geq 0$$

and $tg_1(v,v) + (1-t)g_2(v,v) = 0$
only if $g_1(v,v) = 0$ or $g_2(v,v) = 0$, and in either case this implies $v = 0$.

Lemma: let S be a convex subset of a vector space V.
then if S contains $n+1$ vectors $v_0, v_1, \dots, v_n$, it also contains all points in the n-simplex

$$\{ t_0 v_0 + t_1 v_1 + \dots + t_n v_n \mid t_i \geq 0,\ t_0 + t_1 + \dots + t_n = 1 \}$$

Picture proof:



Proof of theorem: Take trivializing atlas $\{U_\alpha, \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n\}$
Let g_α denote the metric induced on $\pi^{-1}(U_\alpha)$ from the standard metric on $\mathbb{R}^n$ via the fiberwise isomorphism φ_α.
Let $\{\rho_\alpha\}$ denote a partition of unity subordinate to the covering $\{U_\alpha\}$ (Recall $\operatorname{supp} \rho_\alpha \subset U_\alpha$, $0 \leq \rho_\alpha \leq 1$,

$\sum_\alpha \rho_\alpha(x) = 1$) Then $g = \sum_\alpha \rho_\alpha g_\alpha$

defines globally a symmetric bilinear form on $\pi : E \to X$

at each point g_x is in the simplex spanned by the g_α for α such that $x \in U_\alpha$, so it is a metric. ☒

Metrics on the Tangent bundle are the fundamental object of study in Riemannian geometry. For this lecture we will use them to prove a structure theorem for subbundles.

Theorem: Suppose $\pi : E \to X$ admits a metric (always true if X is a manifold, by the above).
Let $F \subset E$ be a subbundle.
Then F is a Whitney summand of E. That is, there exists another subbundle $F' \subset E$ and an isomorphism

$$F \oplus F' \xrightarrow{\sim} E$$

More precisely, choosing a metric g on E, $F' = F^\perp$, the orthogonal complement of F (which is a locally trivial vector bundle over X).

Lemma: let $E \to X$ be a vector bundle with metric g. Then locally at anypoint, E admits an orthonormal frame, that is, a collection of $n = \mathrm{rank}(E)$ local sections $\{s_i(x)\}_{i=1}^n$ such that $g_x(s_i(x), s_j(x)) = \delta_{ij}$

Proof Start with any frame $\tilde{s}_i(x)$ coming from a local trivialization. Just apply Gram-Schmidt:

$$s_1(x) = \tilde{s}_1(x) \Big/ \underbrace{\left(g_x(\tilde{s}_1(x), \tilde{s}_1(x))\right)^{1/2}}_{\text{norm.}}$$

$$s_2(x) = \frac{\tilde{s}_2(x) - g(\tilde{s}_2(x), s_1(x))\, s_1(x)}{\text{norm}}$$

$\vdots$

$\blacksquare$

Proof of theorem: Choose a metric g on $\pi : E \to X$. Define a subset $F_x^\perp \subset E_x$ by $F_x^\perp = \{ v \mid \forall u \in F_x,\ g(u, v) = 0 \}$

$F_x^\perp$ is a linear subspace of E_x. let $F^\perp = \bigcup_{x \in X} F_x^\perp$.
We need to see that $F^\perp$ is a manifold and locally trivial

Choose local orthonormal frames $s_1, \dots, s_k$ for F
($k = \mathrm{rank}\, F$, $n = \mathrm{rank}\, E$) $\quad r_1, \dots, r_n$ for E
In a open set U containing some point x_0

Now consider the $k \times n$ matrix $\left[g(s_i(x_0), r_j(x_0)) \right]_{\substack{i=1\cdots k \\ j=1\cdots n}}$

It has full rank $= k$ so after renumbering we may assume the first k columns are linearly independent

Then in an open set around x_0, first k columns of $[g(s_i(x), r_j(x))]$ are indep.

This means that no combination of $\{s_i\}_{i=1}^k$ is orthogonal to all $\{r_j\}_{j=1}^k$.

This implies that $s_1, s_2, \dots, s_k, r_{k+1}, \dots, r_n$ is a linearly independent set, ie. a frame.

Now apply Gram-Schmidt to $s_1, \dots, s_k, r_{k+1}, \dots, r_n$
The first k vectors $s_1, \dots, s_k$ do not change, they are still an orthonormal frame for F.
The last $n-k$ vectors $s_{k+1}, \dots, s_n$ form an orthonormal frame for $F^\perp$. Thus we get a smooth local trivialization for $F^\perp$ by

$$U \times \mathbb{R}^{n-k} \longrightarrow \bigcup_{x \in U} F_x^\perp$$

$$(x, t_{k+1}, \dots, t_n) \mapsto t_{k+1} s_{k+1}(x) + \dots + t_n s_n(x).$$

The morphism
$$\begin{tikzcd} F \oplus F^\perp \arrow[r] \arrow[dr] & E \arrow[d] \\ & X \end{tikzcd}$$
is obvious, just include F and $F^\perp$ as subbundles

It is also clearly a fiber-wise isomorphism, since F_x and $F_x^\perp$ span E_x and their intersection is zero.

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