# Lecture 4: Smooth functors and constructions with bundles, quotients

Let $\mathcal{V} := \mathrm{Vect}_{\mathbb{R}}^{f.d.}$ denote the category of finite-dimensional $\mathbb{R}$-vector spaces and linear maps.

In linear algebra, we learn various constructions with vector spaces:

- Direct sum $V \oplus W$
- Tensor product $V \otimes W$
- Hom space = linear transformations $\mathrm{Hom}(V, W)$
- dual $V^\vee = \mathrm{Hom}(V, \mathbb{R})$
- Symmetric power e.g. $S^2(V) = V \otimes V / (u \otimes v - v \otimes u)$

All of these natural constructions are <u>functors</u>

tensor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, covariant in both inputs

dual $\vee : \mathcal{V} \longrightarrow \mathcal{V}$, contravariant

Hom : $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, contravariant in first input, covariant in second

Recall that this means we have associated constructions on morphisms which respect composition of morphisms

E.g. for dual: $f: V \to W$ get $f^\vee : W^\vee \to V^\vee$ defined by setting, for $\lambda \in W^\vee = \mathrm{Hom}(W, \mathbb{R})$,

$$f^\vee(\lambda) = (\lambda \circ f) : V \to \mathbb{R}, \text{ an element of } V^\vee = \mathrm{Hom}(V, \mathbb{R})$$

(this is a contravariant example)

We want to promote every such construction in $\mathcal{V}$ to a construction with smooth vector bundles over a fixed but arbitrary smooth manifold $X$.

The rough idea is obvious: Given a functor

$$T : \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{n} \longrightarrow \mathcal{V}$$

and $n$ vector bundles $\xi_i : E_i \to X$ $(i = 1, \ldots, n)$ over the same base $X$.

$$\boxed{\text{Let the fiber of } T(\xi_1, \ldots, \xi_n) \text{ over } x \in X \text{ be } T(\xi_1^{-1}(x), \ldots, \xi_n^{-1}(x)).}$$

But to see that this is a vector bundle, we have to use the fact that $T$ is actually functorial, and in fact we need that $T$ be a <u>smooth functor</u>:

In $\mathcal{V}$, each morphism set $\mathrm{Hom}(V, W)$ is a finite dimensional $\mathbb{R}$ vector space, in particular it has a <u>smooth manifold structure</u>.

The composition map $\circ : \mathrm{Hom}(V, W) \times \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$ is a smooth map of manifolds. (Indeed it is bilinear)

[Thus $\mathcal{V}$ is something like the categorical analogue of a Lie group; a category enriched over manifolds ]

A <u>functor is smooth</u> if the map it yields on morphism sets is smooth.

Eg. for $\otimes$ : $W, X, Y, Z$ vector spaces

$$\begin{aligned} \otimes : \mathrm{Hom}(W, X) \times \mathrm{Hom}(Y, Z) &\longrightarrow \mathrm{Hom}(W \otimes Y, X \otimes Z) \\ (f, g) &\longmapsto f \otimes g \end{aligned}$$

is a smooth (indeed bilinear) map.

So let $T : \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{n} \to \mathcal{V}$ be a smooth functor

let $\xi_i : E_i \to X$ be smooth vector bundles.

let $U \subset X$ be an open set over which all $\xi_i$'s are trivializable, and let $\phi_i : \xi_i^{-1}(U) \to U \times \mathbb{R}^{k_i}$ be a local triv.

denote by $h_i(x) : \xi_i^{-1}(x) \to \mathbb{R}^{k_i}$ the isomorphism of fibers

Define total space $T(E_1, \dots, E_n) = \coprod_{x \in X} T(\xi_1^{-1}(x), \dots, \xi_n^{-1}(x))$

$T(\xi_1, \dots, \xi_n) : T(E_1, \dots, E_n) \to X$ obvious map.

Construct a local trivialization

$$\Phi : T(\xi_1, \dots, \xi_n)^{-1}(U) \longrightarrow U \times T(\mathbb{R}^{k_1}, \dots, \mathbb{R}^{k_n})$$

by sending $T(\xi_1^{-1}(x), \dots, \xi_n^{-1}(x)) \xrightarrow[T(h_1(x), \dots, h_n(x))]{} T(\mathbb{R}^{k_1}, \dots, \mathbb{R}^{k_n})$

Here we <u>really use</u> that $T$ is a functor so that $T(h_1(x), \dots, h_n(x))$ is naturally defined and to see that it is an isomorphism.

Define a smooth structure on $T(\xi_1, \dots, \xi_n)^{-1}(U)$ be declaring $\Phi$ to be a diffeomorphism.

The fact that $T$ is a <u>smooth</u> functor implies that this smooth structure does not depend on the choice of the original trivializations $\phi_i$.

For a different trivialization $\phi_i'$, we have
$$h_i'(x) = A_i(x)\, h(x)$$
where $A_i : U \to GL(k_i, \mathbb{R})$ is smooth.

If $T$ is covariant in index $i$, then
$$T(h_1(x), \ldots, A_i h_i(x), \ldots, h_n(x)) = \underbrace{T(I, \ldots, A_i(x), \ldots, I)} \circ T(h_1(x), \ldots, h_n(x))$$

Since $T$ is smooth, this is a smooth map $U \to GL(T(\mathbb{R}^{k_1}, \ldots, \mathbb{R}^{k_n}))$
Thus the two trivializations $\Phi$ and $\Phi'$ are smoothly related.
Similar thing works for contravariant indices.
This argument also shows that the local trivializations are compatible on overlaps.
This completes the construction.

Now if $\xi : E \to X$, $\eta : F \to X$ are vector bundles, we may freely write
$\xi \otimes \eta : E \otimes F \to X$ tensor product bundle
$\xi^\vee : E^\vee \to X$ dual bundle

For Hom: the symbol $\mathrm{Hom}(E, F)$ potentially ambiguous
it could mean the set of morphisms of bundles
$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

So we will use $\mathcal{H}om(E, F)$ to denote the Hom-bundle, which is a vector bundle over $X$.
$$\mathcal{H}om(E, F)_x = \mathrm{Hom}(E_x, F_x)$$

Other linear algebraic notions

subspace $U \subset V$ $\longrightarrow$ subbundle $F \subset E$

<u>Def</u> A <u>subbundle</u> $F$ of $\xi : E \to X$ is a submanifold $F \subset E$ which is closed under the fiberwise addition and scaler multiplication (so that $F_x = F \cap E_x \subset E_x$ is a subspace) and such that $\xi|_F : F \to X$ is a locally trivial vector bundle.

Observe that the dimensions of the subspaces $F_x = F \cap E_x \subset E_x$ must be locally constant.

Starting with quotients, things start to get trickier.
(In fact, the kernel and cokernel of a morphism
$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$
<u>do not necessarily exist</u> as vector bundles.)

<u>Quotient</u>: Let
$$\begin{tikzcd} F \arrow[r, phantom, "\subset"] \arrow[dr] & E \arrow[d] \\ & X \end{tikzcd}$$
be a subbundle.

In defining the quotient bundle $E/F \longrightarrow X$, the tricky point is to prove local triviality:

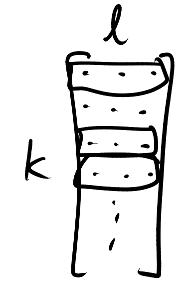
Restricting to a local trivialization of $E$, we may assume that $E$ is trivial. Consider:
$$\begin{tikzcd} F|_U \arrow[r, phantom, "\subset"] \arrow[dr] & U \times \mathbb{R}^k \arrow[d] \\ & U \end{tikzcd}$$

Shrinking the open set $U$ if necessary, we may assume $F$ is trivializable over $U$:
$$\begin{tikzcd} U \times \mathbb{R}^\ell \arrow[dr] & F|_U \arrow[l, "\sim"'] \arrow[r, phantom, "\subset"] \arrow[d] & U \times \mathbb{R}^k \arrow[dl] \\ & U & \end{tikzcd}$$

Thus, the family of subspaces $F_x \subset E_x$, $x \in U$, is decribled with respect to these local trivializations by a smooth family of maps $B(x): \mathbb{R}^l \to \mathbb{R}^k$

satisfying the condition that $B(x)$ is always injective. $B(x)$ is thus a $k \times l$ matrix with full column rank $(=l)$.

We need to use some serious linear algebra to construct a complement to $F_x \subset E_x$:




Since $B(x)$ has rank $l$, there is some $l \times l$ minor determinant that does not vanish. Since the set where a particular minor does not vanish is open, by shrinking $U$, we may assume that a certain minor consisting of rows $i_1, \dots, i_l$ does not vanish throughout $U$.

Here $\{i_1, \dots, i_l\} \subset \{1, \dots, k\}$ is a subset.
Let $\mathbb{R}^l_{i_1 \cdots i_l} \subset \mathbb{R}^k$ denote the subspace where all coordinates other than $x_{i_1} \cdots x_{i_l}$ are zero.

- Then $F_x = \text{image } B(x)$ maps isomorphically onto $\mathbb{R}^l_{i_1 \cdots i_l}$ under orthogonal projection, for all $x \in U$
- Hence $(\mathbb{R}^l_{i_1 \cdots i_l})^\perp \subset \mathbb{R}^k$ is complementary to $F_x$ $\forall x \in U$!
- Thus $U \times (\mathbb{R}^l_{i_1 \cdots i_l})^\perp \longrightarrow U \times \mathbb{R}^k \longrightarrow \coprod_{x \in U} \mathbb{R}^k / \text{image } B(x)$

is a local trivialization of the quotient bundle.