

### Lecture 3: Morphisms of various kinds, Pull back, Whitney direct sum

Let  $E_1, E_2$  be two smooth vector bundles  
 $\xi_1 \downarrow \xi_2 \downarrow$   
 $X_1, X_2$   
 over smooth base manifolds  $X_1, X_2$

Notation: A vector bundle is an example of a diagram of spaces. We will abbreviate a vector bundle  $\xi: E \rightarrow X$  either as  $\xi$  (the name of the bundle projection) or  $E$  (the name of the total space).

Def A morphism from  $\xi_1$  to  $\xi_2$  is a pair of smooth maps  $(f, g)$ :

$$f: X_1 \rightarrow X_2 \quad g: E_1 \rightarrow E_2$$

Such that (1)  $E_1 \xrightarrow{g} E_2$  commutes  
 $\xi_1 \downarrow \downarrow \xi_2$  i.e.,  $f \circ \xi_1 = \xi_2 \circ g$   
 $X_1 \xrightarrow{f} X_2$

(2) The map  $g|_{\xi_1^{-1}(x)}: \xi_1^{-1}(x) \rightarrow \xi_2^{-1}(f(x))$

is a linear map.

(Observe that (1) implies  $g$  maps the fiber over  $x$  into the fiber over  $f(x)$ )

We say  $g$  is a morphism over the base map  $f$ .  
 (or covering the map  $f$ .)

A special case is when  $X_1 = X_2$  and  $f = \text{id}$

Def if  $\xi_1: E_1 \rightarrow X$  and  $\xi_2: E_2 \rightarrow X$  are two vector bundles over a single base  $X$ , a morphism over  $X$  from  $\xi_1$  to  $\xi_2$  is a map  $g: E_1 \rightarrow E_2$

such that (1) 
$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ \xi_1 \downarrow & & \downarrow \xi_2 \\ & X & \end{array} \quad \xi_2 \circ g = \xi_1$$
 commutes

(2)  $g|_{\xi_1^{-1}(x)}: \xi_1^{-1}(x) \rightarrow \xi_2^{-1}(x)$  is linear.

A morphism is called fiberwise injective / fiberwise surjective / fiberwise isomorphism if the map on fibers has the corresponding property whenever it is defined. Note that the base map need not be injective / surjective / bijective.

Trivial example: let  $\{pt\}$  denote the 0-dimensional manifold with one point. Let  $\xi: E \rightarrow X$  be a rank  $k$  vector bundle. Choose a point  $x \in X$ , and an isomorphism  $\phi: \mathbb{R}^k \rightarrow E_x$

Then the diagram

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{\phi} & E \\ \downarrow & & \downarrow \xi \\ \{pt\} & \longrightarrow & X \\ & p^+ \longmapsto & x \end{array}$$

is a morphism of vector bundles which is a fiberwise isomorphism. But it is not an isomorphism of vector bundles as long as  $X$  has more than one point.

A "morphism of vector bundles which is a fiberwise isomorphism" is also called a bundle map (possibly confusingly).

Pullback: Let  $\xi: E \rightarrow X$  be a smooth vector bundle and let  $f: Y \rightarrow X$  be a smooth map

So we have a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array}$$

we will complete it to a bundle map diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ f^*\xi \downarrow & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array}$$

I.E., we will construct a smooth manifold  $f^*E$ , a smooth map  $f^*E \rightarrow Y$ , such that  $f^*E \rightarrow Y$  is a smooth vector bundle, and a map  $f^*E \rightarrow E$  which is a fiberwise isomorphism

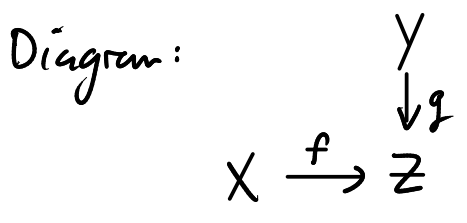
The bundle  $f^*\xi: f^*E \rightarrow Y$  is called the pullback of  $\xi: E \rightarrow X$  by  $f: Y \rightarrow X$

Roughly speaking, we define the fiber  $(f^*E)_y$  over  $y \in Y$  to be equal to the fiber  $E_{f(y)}$  of the original bundle over the image  $f(y) \in X$ .

A rather neat way to do this is with the general notion of a fiber product.

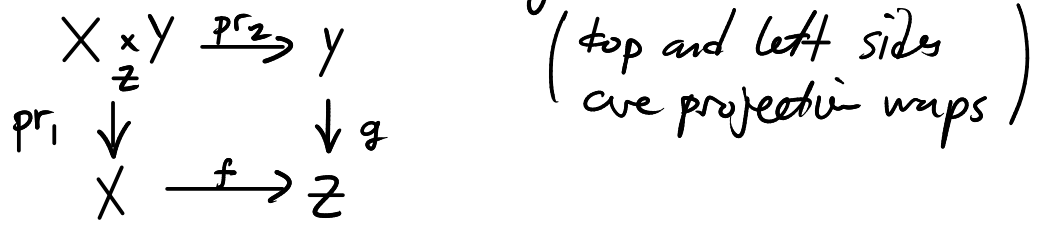
Fiber product (of sets)

Let  $X, Y, Z$  be sets,  
 $f: X \rightarrow Z, g: Y \rightarrow Z$  maps

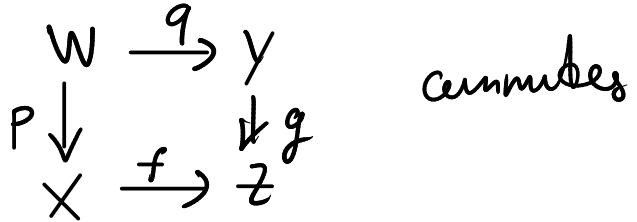


The fiber product of  $X$  and  $Y$  over  $Z$  (w.r.t.  $f, g$ ) is  
 $X \times_Z Y = \{ (x, y) \in X \times Y \mid f(x) = g(y) \}$

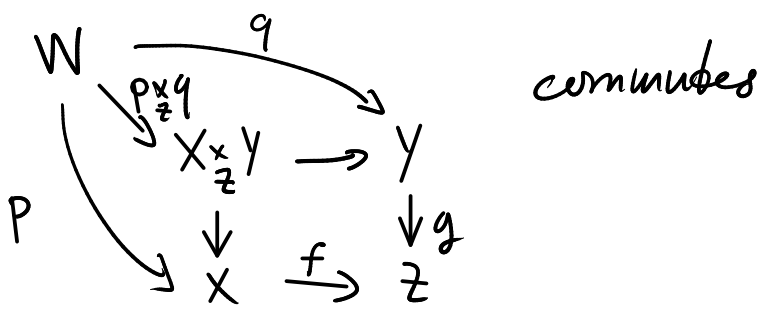
(1)  $X \times_Z Y$  fits into a commutative diagram



(2) (Universal property) If  $W$  is a set and  $p: W \rightarrow X$  and  $q: W \rightarrow Y$  are maps such that



Then there is a unique map such that  $p \times q: W \rightarrow X \times_Z Y$



Indeed:  $p \times q$  is necessarily the map  $w \mapsto (p(w), q(w))$  and this map has the desired property.

(3) The projection  $\text{pr}_2 : X \times_Z Y \rightarrow Y$  maps  $\text{pr}_1^{-1}(x)$  bijectively to  $g^{-1}(f(x))$

Now suppose  $X, Y, Z$  are manifolds and  $f, g$  are smooth maps. When is  $X \times_Z Y$  a manifold?

Def  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  are transverse if  $\forall z \in Z, \forall x \in f^{-1}(z), \forall y \in g^{-1}(z)$

$$Df(T_x X) + Dg(T_y Y) = T_z Z$$

Thm If  $f$  and  $g$  are transverse then  $X \times_Z Y$  is a smooth submanifold of  $X \times Y$ .

Observe: If  $g$  is a submersion, then  $f$  and  $g$  are always transverse!

Observe: If  $\xi: E \rightarrow X$  is a vector bundle, then  $\xi$  is a submersion.

Construction of pullback: given  $\left. \begin{array}{c} E \\ \downarrow \xi \\ Y \xrightarrow{f} X \end{array} \right\}$  let  $f^*E = Y \times_X E$  be the fiber product over  $X$

Then  $f^*E$  is a smooth manifold, and we have a diagram of

smooth maps  
by property (1)

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ f^*\xi \downarrow & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array}$$

By property (3) The map  $f^*E \xrightarrow{g} E$  maps  $(f^*\xi)^{-1}(y)$  bijectively onto  $\xi^{-1}(f(y))$ , so the fibers of  $f^*\xi$  inherit vector space structures from the fibers of  $\xi$ .

It remains to show  $f^*\xi: f^*E \rightarrow Y$  is locally trivial  
 Let  $U \subset X$  be an open set and  $\phi: \xi^{-1}(U) \rightarrow U \times \mathbb{R}^k$   
 a local trivialization. Let  $h: \xi^{-1}(U) \rightarrow \mathbb{R}^k$  denote  
 the  $\mathbb{R}^k$ -component of  $\phi$ .

Over the open set  $f^{-1}(U) \subset Y$  define local trivialization

$$\begin{aligned} (f^*\xi)^{-1}(f^{-1}(U)) &\longrightarrow f^{-1}(U) \times \mathbb{R}^k \\ q &\longmapsto (f^*\xi(q), h(q)) \end{aligned}$$

This completes the construction of the pullback.

Application: Whitney sum (direct sum) of vector bundles

Let  $\xi: E \rightarrow X$  and  $\eta: F \rightarrow X$  be two vector bundles  
 over the same base  $X$ .

We want a vector bundle  $\xi \oplus \eta: E \oplus F \rightarrow X$  whose  
 fibers are the direct sums of the corresponding fibers of  $E$  and  $F$ .

There is obviously a vector bundle  $\xi \times \eta: E \times F \rightarrow X \times X$   
 where we take cartesian product of the base spaces and total spaces.  
 (The fiber over  $(x_1, x_2)$  is the sum  $E_{x_1} \oplus F_{x_2}$ .)

Let  $\delta: X \rightarrow X \times X$   $\delta(x) = (x, x)$  be the diagonal map.

Define the Whitney sum  $\xi \oplus \eta: E \oplus F \rightarrow X$

to be the pullback  $\delta^*(\xi \times \eta): \delta^*(E \times F) \rightarrow X$  of the  
 Cartesian product by the diagonal map  $\delta$ .