# Lecture 3: Morphisms of various kinds, Pull back, Whitney direct sum

Let $\begin{matrix} E_1 \\ \xi_1 \downarrow \\ X_1 \end{matrix}$ $\begin{matrix} E_2 \\ \xi_2 \downarrow \\ X_2 \end{matrix}$ be two smooth vector bundles over smooth base manifolds $X_1, X_2$

<u>Notation</u>: A vector bundle is an example of a <u>diagram</u> of spaces. We will abreviate a vector bundle $\xi : E \to X$ either as $\xi$ (the name of the bundle projection) or $E$ (the name of the total space).

<u>Def</u> A <u>morphism</u> from $\xi_1$ to $\xi_2$ is a pair of smooth maps $(f, g)$:

$$f : X_1 \longrightarrow X_2 \qquad g : E_1 \longrightarrow E_2$$

Such that (1)
$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "\xi_1"'] & E_2 \arrow[d, "\xi_2"] \\ X_1 \arrow[r, "f"'] & X_2 \end{tikzcd}$$
commutes i.e., $f \circ \xi_1 = \xi_2 \circ g$

(2) The map $g|_{\xi_1^{-1}(x)} : \xi_1^{-1}(x) \longrightarrow \xi_2^{-1}(f(x))$

is a linear map.

(Observe that (1) implies $g$ maps the fiber over $x$ into the fiber over $f(x)$)

We say $g$ is a morphism over the base map $f$. (or covering the map $f$.)

A special case is when $X_1 = X_2$ and $f = id$

<u>Def</u> if $\xi_1 : E_1 \to X$ and $\xi_2 : E_2 \to X$ are two vector bundles over a single base $X$, a <u>morphism over $X$</u> from $\xi_1$ to $\xi_2$ is a map $g : E_1 \to E_2$

Such that (1)

$$\begin{tikzcd} E_1 \arrow[rr, "g"] \arrow[dr, "\xi_1"'] & & E_2 \arrow[dl, "\xi_2"] \\ & X & \end{tikzcd}$$

commutes $\xi_2 \circ g = \xi_1$

(2) $g|_{\xi_1^{-1}(x)} : \xi_1^{-1}(x) \longrightarrow \xi_2^{-1}(x)$ is linear.

A morphism is called fiberwise injective / fiberwise surjective / fiberwise isomorphism if the map on fibers has the corresponding property whenever it is defined. Note That the base map <u>need not</u> be injective/surjective/bijective.

<u>Trivial example</u>: let $\{pt\}$ denote the 0-dimensional manifold with one point. Let $\xi : E \to X$ be a rank $k$ vector bundle. Choose a point $x \in X$, and an isomorphism $\phi : \mathbb{R}^k \to E_x$

Then the diagram

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "\phi"] \arrow[d] & E \arrow[d, "\xi"] \\ \{pt\} \arrow[r] & X \end{tikzcd}$$

$$pt \longmapsto x$$

is a morphism of vector bundles which is a <u>fiberwise isomorphism</u> But it is <u>not an isomorphism</u> of vector bundles as long as $X$ has more than one point.

A "morphism of vector bundles which is a fiberwise isomorphism" is also called a <u>bundle map</u> (possibly confusingly).

<u>Pullback</u>: Let $\xi : E \to X$ be a smooth vector bundle and let $f : Y \to X$ be a smooth map

So we have a diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array}$$

we will complete it to a bundle map diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ f^*\xi \downarrow & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array}$$

I.E., we will construct a smooth manifold $f^*E$, a smooth map $f^*E \to Y$, such that $f^*E \to Y$ is a smooth vector bundle, and a map $f^*E \to E$ which is a fiberwise isomorphism

The bundle $f^*\xi : f^*E \to Y$ is called the <u>pullback</u> of $\xi : E \to X$ by $f : Y \to X$

Roughly speaking, we define the fiber $(f^*E)_y$ over $y \in Y$ to be equal to the fiber $E_{f(y)}$ of the original bundle over the image $f(y) \in X$.

A rather neat way to do this is with the general notion of a <u>fiber product</u>.

## Fiber product (of sets)

Let $X, Y, Z$ be sets, $f : X \to Z$, $g : Y \to Z$ maps

Diagram:
$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

The fiber product of $X$ and $Y$ over $Z$ (w.r.t. $f, g$) is
$$X \times_Z Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \}$$

(1) $X \times_Z Y$ fits into a commutative diagram
$$\begin{tikzcd} X \times_Z Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$
(top and left sides are projection maps)

(2) (Universal property) If $W$ is a set and $p : W \to X$ and $q : W \to Y$ are maps such that
$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$
commutes

Then there is a unique map $p \times_Z q : W \to X \times_Z Y$ such that
$$\begin{tikzcd} W \arrow[dr, "p \times_Z q"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$
commutes

<u>Indeed</u>: $p \times_Z q$ is necessarily the map $w \mapsto (p(w), q(w))$ and this map has the desired property.

(3) The projection $pr_2 : X \times_Z Y \to Y$ maps $pr_1^{-1}(x)$ bijectively to $g^{-1}(f(x))$

Now suppose $X, Y, Z$ are manifolds and $f, g$ are smooth maps. When is $X \times_Z Y$ a manifold?

<u>Def</u> $f : X \to Z$ and $g : Y \to Z$ are <u>transverse</u> if $\forall z \in Z, \forall x \in f^{-1}(z), \forall y \in g^{-1}(z)$

$$Df(T_x X) + Dg(T_y Y) = T_z Z$$

<u>Thm</u> If $f$ and $g$ are transverse then $X \times_Z Y$ is a smooth submanifold of $X \times Y$.

Observe: If $g$ is a submersion, then $f$ and $g$ are always transverse!

Observe: If $\xi : E \to X$ is a vector bundle, then $\xi$ is a submersion.

<u>Construction of pullback</u>: given

$$\left\{ \begin{array}{ccc} & & E \\ & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array} \right\}$$

let $f^*E = Y \times_X E$ be the fiber product over $X$

Then $f^*E$ is a smooth manifold, and we have a diagram of smooth maps by property (1)

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ {\scriptstyle f^*\xi}\downarrow & & \downarrow \xi \\ Y & \xrightarrow{f} & X \end{array}$$

By property (3) The map $f^*E \xrightarrow{g} E$ maps $(f^*\xi)^{-1}(y)$ bijectively onto $\xi^{-1}(f(y))$, so the fibers of $f^*\xi$ inherit vector space structures from the fibers of $\xi$.

It remains to show $f^*\xi : f^*E \to Y$ is locally trivial

Let $U \subset X$ be an open set and $\phi : \xi^{-1}(U) \to U \times \mathbb{R}^k$ a local trivialization. Let $h : \xi^{-1}(U) \to \mathbb{R}^k$ denote the $\mathbb{R}^k$-component of $\phi$.

Over the open set $f^{-1}(U) \subset Y$ define local trivialization

$$(f^*\xi)^{-1}(f^{-1}(U)) \longrightarrow f^{-1}(U) \times \mathbb{R}^k$$
$$q \longmapsto (f^*\xi(q),\ h(g(q)))$$

This completes the construction of the pullback.

<u>Application</u>: Whitney sum (direct sum) of vector bundles

Let $\xi : E \to X$ and $\eta : F \to X$ be two vector bundles over the same base $X$.

We want a vector bundle $\xi \oplus \eta : E \oplus F \to X$ whose fibers are the direct sums of the corresponding fibers of $E$ and $F$.

There is obviously a vector bundle $\xi \times \eta : E \times F \to X \times X$ where we take cartesian product of the base spaces and total spaces. (The fiber over $(x_1, x_2)$ is the sum $E_{x_1} \oplus F_{x_2}$.)

Let $\delta : X \to X \times X$ $\delta(x) = (x, x)$ be the diagonal map.

Define the <u>Whitney sum $\xi \oplus \eta : E \oplus F \to X$</u>

to be the pullback $\delta^*(\xi \times \eta) : \delta^*(E \times F) \to X$ of the Cartesian product by the diagonal map $\delta$.