Grassmannians and Classification of vector bundles.

The **Grassmannian** of k-planes in n-space is the set

$$G_k(\mathbb{R}^n) = \{ V \mid V \subset \mathbb{R}^n \text{ is a linear subspace of dim } k \}$$

- $G_k(\mathbb{R}^n)$ admits the structure of a smooth manifold of dimension $k(n-k)$. The tangent space at the point $V \in G_k(\mathbb{R}^n)$ is $T_V G_k(\mathbb{R}^n) \simeq \mathrm{Hom}(V, V^\perp)$.

- The case $k=1$ is projective space: $G_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$

- The correspondence $V \leftrightarrow V^\perp$ induces a diffeomorphism $G_k(\mathbb{R}^n) \cong G_{n-k}(\mathbb{R}^n)$

- There are two tautological bundles over $G_k(\mathbb{R}^n)$
The subspace bundle $S_k^n \to G_k(\mathbb{R}^n)$

$$S_k^n = \{ (V, x) \in Gr_k(\mathbb{R}^n) \times \mathbb{R}^n \mid x \in V \}$$ whose fiber at V is V itself.

Let $\underline{\mathbb{R}}^n = Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$ be the trivial bundle.
Then $S_k^n \to \underline{\mathbb{R}}^n$ is a subbundle, and the Quotient is a vector bundle of rank $n-k$: Q_k^n = Quotient bundle.
The fiber of Q_k^n at V is $\mathbb{R}^n/V$. There is an exact sequence of vector bundles over $G_k(\mathbb{R}^n)$:

$$0 \to S_k^n \to \underline{\mathbb{R}}^n \to Q_k^n \to 0$$

Recall pullback of vector bundles via fiber product

$f: M \to N$ smooth map $\qquad \pi: E \to N$ vector bundle

$$\begin{array}{ccc} f^*E = \{ (x,v) \in M \times E \mid f(x) = \pi(v) \} \\ \downarrow \qquad\qquad\qquad \overline{\downarrow} \\ M \quad \ni \quad x \end{array}$$

(The fiber of f^*E at $x \in M$ is the fiber of E at $f(x) \in N$)

<u>Key idea</u>: Let M be a manifold. For every vector bundle $\pi: E \to M$ of rank r, there is a map $f: M \to G_{n-r}(\mathbb{R}^n)$ such that $f^* Q^n_{n-r} \cong E$ as vector bundles over M, where n is sufficiently large. (and one may need to take $n = \infty$ — more on this later.)

- Note that Q^n_{n-r} has rank r no matter what n is.

The idea is that every vector bundle is isomorphic to a pullback of a tautological quotient bundle over a Grassmannian. In this sense the Grassmannian and its tautological bundles are a <u>universal</u> family of objects.

Of course two maps $f, g: M \to G_{n-r}(\mathbb{R}^n)$ may determine isomorphic vector bundles. In fact we have:

<u>Thm</u> (homotopy invariance of pullback) If $f, g: M \to N$ are homotopic maps, and $\pi: E \to M$ is a vector bundle, then f^*E and g^*E are isomorphic as vector bundles over M.

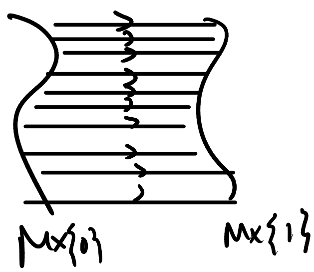
<u>Proof</u> let $h: M \times [0,1] \to N$ be a homotopy: $h(x,0) = f(x)$
$h(x,1) = g(x)$

And consider $\begin{array}{c} h^*E \\ \downarrow \\ M \times [0,1] \end{array}$. There are two inclusions:

$$\left\{ \begin{array}{ll} i_0 : M \to M \times [0,1] & x \mapsto (x,0) \\ i_1 : M \to M \times [0,1] & x \mapsto (x,1) \end{array} \right\}$$

and $\left\{ \begin{array}{l} f = h \circ i_0 \\ g = h \circ i_1 \end{array} \right\}$ so $\left\{ \begin{array}{l} f^*E = i_0^*(h^*E) \\ g^*E = i_1^*(h^*E) \end{array} \right\}$

Pick a connection on $h^*E \to M \times [0,1]$. The parallel transport along the segments $t \mapsto (x,t)$ for fixed x gives isomorphisms $(h^*E)_{(x,0)} \simeq (h^*E)_{(x,1)}$, and a vector bundle isomorphism $i_0^*(h^*E) \simeq i_1^*(h^*E)$



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<u>Corollary</u> Any vector bundle on a contractible manifold is trivial.

<u>Proof</u> since the maps $M \xrightarrow{id} M$ and $M \xrightarrow{f} \{x\} \subset M$ are homotopic we have $E = id^*E \simeq f^*E = M \times E_x$

Thus if $Vect_r(M)$ denotes the set of isomorphism classes of rank r vector bundles over M, and $[M,N]$ denotes the set of homotopy classes of maps from M to N, we have a map

$$\begin{array}{ccc} [M, G_{n-r}(\mathbb{R}^n)] & \longrightarrow & Vect_r(M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [f] & \longmapsto & [f^* Q^n_{n-r}] \end{array}$$

We want to say that for n sufficently large, this map is onto (but in general we need to take $n = \infty$).
We would also like this map to be a bijection, but we need some hypotheses to be able to do this for finite n.

<u>Proposition</u>: Suppose that $\pi : E \to M$ (rank r) admits n sections $s_1, \dots, s_n$ such that $s_1(x), \dots, s_n(x)$ span E_x for every $x \in M$. Then there is a map $f: M \to G_{n-r}(\mathbb{R}^n)$ such that $E \simeq f^* Q^n_{n-r}$

<u>Proof</u> We must associate to $x \in M$ a subspace of $\mathbb{R}^n$ of dimension $n-r$
The map $\mathbb{R}^n \longrightarrow E_x$

$$(a_1, \dots, a_n) \longmapsto a_1 s_1(x) + \cdots + a_n s_n(x)$$

is, by hypothesis, a surjective linear map from $\mathbb{R}^n$ to a vector space of dimension r. Therefore its kernel $V_x \subset \mathbb{R}^n$ is a subspace of dimension $n-r$, and the quotient $\mathbb{R}^n / V_x$ maps isomorphically onto E_x.

So we set

$$\begin{aligned} f : M &\longrightarrow G_{n-r}(\mathbb{R}^n) \\ x &\longmapsto V_x \end{aligned}$$

and the isomorphisms $\mathbb{R}^n / V_x \to E_x$ fit together to give a bundle isomorphism $f^* Q^n_{n-r} \to E$.

<u>Def</u> A good cover of M is a covering $\{U_i\}$ by open subsets such that each intersection $U_{i_1} \cap \cdots \cap U_{i_s}$ is either empty or diffeomorphic to $\mathbb{R}^d$ ($d = \dim M$). M has finite type if admits a good cover with finitely many open subsets.

<u>Thm</u> Let M be finite type, admitting a good cover by k subsets. Let $\pi: E \to M$ be a rank r vector bundle.

(i) If $n \geq kr$, then there is a map $f: M \to G_{n-r}(\mathbb{R}^n)$ such that $E \simeq f^* Q^n_{n-r}$

(ii) If additionally $n \geq \dim M + r - 1$, any two such maps $M \to G_{n-r}(\mathbb{R}^n)$ are homotopic.

<u>Cor</u> For $n \geq \max\{kr, \dim M + r - 1\}$, there is a bijection

$$[M, G_{n-r}(\mathbb{R}^n)] \cong \mathrm{Vect}_r(M)$$

The proof of (i) uses the previous proposition.
Let $M = U_1 \cup \cdots \cup U_k$ be a good cover.
Since U_i is contractible $\begin{array}{c}\pi^{-1}(U_i) \subset E \\ \downarrow \\ U_i\end{array}$ is trivial.

Let $s_{i,1}, s_{i,2}, \ldots, s_{i,r}$ be trivializing sections. Let $\{\rho_i\}_{i=1}^k$ be a partition of unity subordinate to the covering $\{U_i\}_{i=1}^k$.

Then $\{\rho_i s_{i,j}\}_{i=1,\ldots,k,\ j=1,\ldots,r}$ are rk global sections

And they span E_x at every point. (Where $\rho_i \neq 0$, $\{\rho_i s_{i1}, \ldots, \rho_i s_{ir}\}$ span). Thus, for $n \geq kr$, E admits n global sections satisfying the hypotheses of the proposition.

The proof of part (ii) is more involved and is omitted. ∎

For the purposes of building an abstract theory, it is useful to eliminate the auxiliary parameter n from the story.

We have maps $\mathbb{R}^n \to \mathbb{R}^{n+k}$

$$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0,\dots,0)$$

$$\begin{array}{ccc} G_{n-r}(\mathbb{R}^n) & \to & G_{n+k-r}(\mathbb{R}^{n+k}) \\ \Psi & & \Psi \\ V & \longmapsto & V \oplus \mathbb{R}^k \end{array}$$

and corresponding bundle maps $Q^n_{n-r} \to Q^n_{n+k-r}$

We must take the limit as $n \to \infty$ of $G_{n-r}(\mathbb{R}^n)$ using these maps to see how each space sits inside the next, and taking the union of all of them. The result is $G^\infty_{\infty-r}$ whose points correspond to codimension r subspaces of $\mathbb{R}^\infty$. There is a quotient bundle $Q^r_{\infty-r}$ of rank r. Then we have for any manifold.

$$\begin{array}{ccc} [M, G^\infty_{\infty-r}] & \simeq & \mathrm{Vect}_r(M) \text{ is a bijection} \\ [f] & \longmapsto & [f^* Q^\infty_{\infty-r}] \end{array}$$

For this reason $G^\infty_{\infty-r}$ is called the classifying space for real rank r vector bundles, and $Q^\infty_{\infty-r}$ is called the universal real rank r vector bundle.

A completely analogous story works for complex vector bundles, we consider $G_{n-r}(\mathbb{C}^n)$ and $G^\infty_{\infty-r}(\mathbb{C})$.

Characteristic classes are just cohomology classes of the classifying space. For if $c \in H^i(G^\infty_{\infty-r})$ then one can set $c(E) = f^*c$ where $f: M \to G^\infty_{\infty-r}$ is a classifying map of E: $f^* Q^\infty_{\infty-r} \simeq E$.

In this way, the Chern classes correspond to the generators of the cohomology ring of $G^{\infty}_{\infty-r}(\mathbb{C})$.

The mod 2 cohomology of $G^{\infty}_{\infty-r}(\mathbb{R})$ gives rise to the Stiefel-Whitney classes, but since they are mod 2 classes, they cannot be described in terms of differential forms.