Thom class and Euler Class

M manifold, not necessarily compact.

Recall: compactly supported forms $\Omega^\bullet_c(M)$
compactly supported cohomology $H^\bullet_c(M)$

So we have two cohomologies associated to M $H^\bullet(M)$, $H^\bullet_c(M)$
If we pick an orientation for M (assuming one exists) we have a pairing $H^k(M) \times H^{n-k}_c(M) \to \mathbb{R}$

$$(\alpha, \beta) \longrightarrow \int_M \alpha \wedge \beta$$

The Poincaré duality theorem says that this induces an isomorphism $H^k(M) \simeq H^{n-k}_c(M)^\vee$

Now let $\pi: E \to M$ be a real vector bundle.
* Assume that both M and E are orientable and choose orientations for each. [In particular, this implies that the fibers of E can be oriented in a consistent manner.]

Since E is a vector bundle over M, it is homotopy equivalent to M, and we have an isomorphism
$$H^\bullet(E) \simeq H^\bullet(M)$$
[NB: compactly supported cohomology is not homotopy invariant]

Suppose that M has dimension d and E has rank r.
Then E has dimension d+r as a manifold.

We have
$$\begin{aligned} H^k_c(E) &\simeq H^{d+r-k}(E)^\vee && \text{by Poincaré duality on } E \\ &\simeq H^{d+r-k}(M)^\vee && \text{by homotopy invariance} \\ &\simeq H^{k-r}_c(M) && \text{by Poincaré duality for } M. \end{aligned}$$

If in addition M is compact, we have $H^\bullet_c(M) \simeq H^\bullet(M)$, so we get

Thom isomorphism: For M compact, E & M oriented:
$$H^k_c(E) \cong H^{k-r}(M)$$

Thom class: $\tau(E) \in H^r_c(E)$ is the class corresponding to $1 \in H^0(M)$ under the Thom isomorphism. We emphasise that τ depends on the choice of orientations

Euler class: $e(E) \in H^r(M)$ is obtained from $\tau(E)$ by pulling back under a section $s: M \to E$.
$$e(E) = s^* \tau(E)$$
Note that E always has a section namely the zero section, and any two sections are homotopic via $s_t = t s_1 + (1-t) s_0$ so $e(E)$ is well-defined.

Euler number: In the special case that the rank (r) equals the dimension of the base (d) we can integrate the Euler class over the whole of M:

$$\chi(E) = \int_M e(E)$$

Theorem: If M is an oriented compact manifold, then
$$\int_M e(TM) = \chi(M)$$ ← topological Euler characteristic

Also in the case $r=d$, we have that $\chi(E)$ equals the oriented intersection number of a general section $s: M \to E$ with the zero section. This in particular means that

<u>Prop</u> If $\pi: E \to M$ admits a nowhere vanishing section, then $\chi(E) = 0$.

To understand the Euler class in terms of differential forms, Let's go back to the Thom isomorphism.

$$\pi_*: H^k_c(E) \xrightarrow{\sim} H^{k-r}(M)$$

This map is called integration along the fiber. In a local trivialization: coords $x^1, \dots, x^d$ on $U \subset M$, $t^1, \dots, t^r$ on fiber, we can write any k-form ω as

$$\omega = \alpha \wedge dt^1 \wedge \dots \wedge dt^r \qquad (\alpha \in \Omega^{k-r}(E))$$
$$+ \text{(terms that involve } < r \ dt\text{'s)}$$

We define π_* to be zero on terms involving $< r$ dt's and

$$\pi_*(\alpha \wedge dt^1 \wedge \dots \wedge dt^r) = \int_{\vec{t} \in \mathbb{R}^r} \alpha \, dt^1 \cdots dt^r \in \Omega^{k-r}(M)$$

- This improper integral converges because α will always be compactly supported in the t-directions.
- Note that $\pi_*(\omega) = 0$ automatically if $k < r$

We have $\pi_*: \Omega^k_c(E) \to \Omega^{k-r}(M)$ is a chain map $\pi_* d = d \pi_*$ so π_* induces a map $H^k_c(E) \to H^{k-r}(M)$.

The <u>projection formula</u> says $\quad \pi_*((\pi^*\theta) \wedge \omega) = \theta \wedge \pi_* \omega$

where $\quad \theta \in \Omega^i(M) \quad \omega \in \Omega^j_c(E)$

A Thom form is some $\tau \in \Omega^r_c(E)$, $d\tau = 0$, that integrates to 1 along every fiber: $\pi_* \tau = 1 \in \Omega^0(M)$. The Thom isomorphism implies that any two Thom forms are cohomologous.

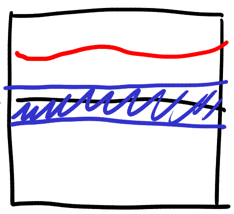
We can now prove that $e(E)$ is the obstruction to the existence of a nowhere vanishing section.

<u>Prop</u> If $\pi: E \to M$ admits a nowhere vanishing section $s: M \to E$, then $e(E) = 0$.

<u>Proof</u>: Let $s: M \to E$ be nowhere vanishing section. Then for $R \gg 0$, the image of Rs is "far" from the zero section. On the other hand, a Thom form τ is compactly supported so its support is near the zero section.

For $R \gg 0$, $\mathrm{im}(Rs) \cap \mathrm{supp}\,\tau = \emptyset$

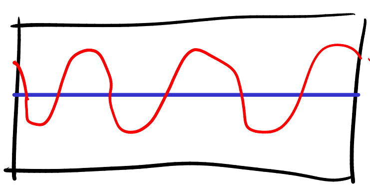
so $e(E) = [(Rs)^* \tau] = [0] = 0$

0-sectn 

(Quantitatively: pick metric on E
$R_1 = \max\{\|v\| \mid v \in \mathrm{Supp}\,\tau\} < \infty$
$R_2 = \min\{\|s(x)\| \mid x \in M\} > 0$
Let $R > R_1/R_2$) ∎

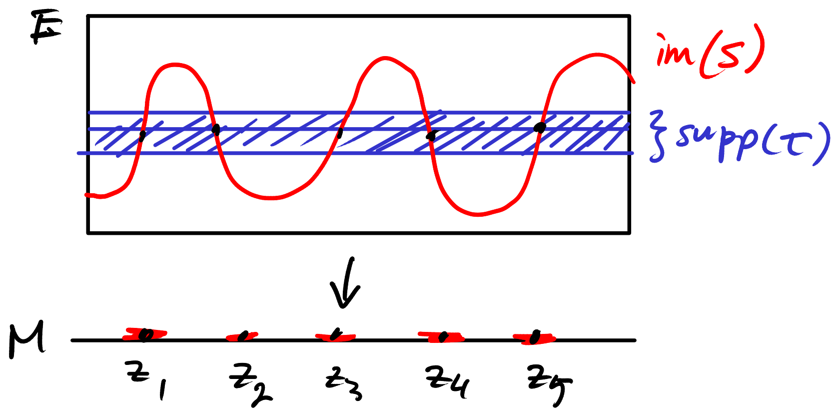
If $d = r$, we can show that $\chi(E) = \int_M e(E)$ is the oriented intersection number of a general transverse section with the zero section.

Let $s: M \to E$ be a transverse section. It intersects the zero section in a finite set of points

0-section 

What we are computing is $\int_M s^*\tau$, where τ is a Thom form

Again, we can assume that the support of the Thom form is close to the zero section, so that $\mathrm{im}(s)$ only intersects $\mathrm{supp}(\tau)$ in small balls around the zeros of s.



So the integral reduces to a sum of integrals over the balls around the zeros of s.
Next, we use local trivializations in each of these balls, yielding local coordinates $(x,t) \in B_i \times \mathbb{R}^d$ (where x are coords on base, t coords on fiber).
Since the Thom class is uniquely determined by the condition that it integrate to 1 on each fiber, we are free to assume that in the local coords, τ doesn't involve x:

$$\tau = f(t)\, dt^1 \wedge \cdots \wedge dt^n \qquad \text{for } (x,t) \in B_i \times \mathbb{R}^d$$

where $f(t)$ is a bump function such that

$$\int_{\mathbb{R}^n} f(t)\, dt^1 \cdots dt^n = 1$$

The section s looks like $x \mapsto (x, t(x))$ in local trivialization.
By invariance of the integral of a differential form,

$$\int_{B_i} s^*(f(t)\, dt^1 \wedge \cdots \wedge dt^n) = \pm 1$$

$B_i \longrightarrow \mathbb{R}^d$

depending on whether the map $x \mapsto t(x)$ respects the chosen orientations. This is why the total is the <u>oriented</u> intersection number ∎

<u>Equality of Euler Characteristics</u>: Let M be an oriented manifold. Let $V \in \Gamma(TM)$ be a vector field, that is a section of the tangent bundle. Suppose that V is transverse to the zero section. Then let $I(V)$ denote the oriented intersection number (also called index).

It follows from the preceding that $I(V) = \int_M e(TM)$

In particular $I(V)$ <u>does not depend</u> on the choice of V!

On the other hand, there are vector fields V such that $I(V)$ is the topological Euler characteristic:

$$\chi(M) = \sum_i (-1)^i (\#\text{simplices of dim } i)$$

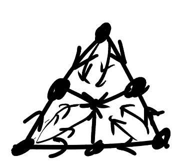
Thus $$\chi(M) = I(V) = \int_M e(TM) \quad !$$

<u>Example construction</u>. Triangulate M. Parts are <u>simplices</u>

Construct a vector field this way

on 0-simplex • just zero

on 1-simplex  zero in the midpoint

on 2-simplex  zero in the center.

continue to higher dimensions. The vector field so constructed will have one zero for each simplex, and one finds that a simplex of dimension i contributes $(-1)^i$ to $I(V)$.