Lecture 23

Some calculations of topological invariants for complex manifolds.

Def An almost complex manifold is a manifold together with the structure of a complex vector bundle on its tangent bundle. That is, there is a complex vector bundle $E \to M$ and an isomorphism of real vector bundles

$$\begin{tikzcd} E_{\mathbb{R}} \arrow[rr, "\sim"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

where $E_{\mathbb{R}}$ is the underlying real vector bundle of E.

It is equivalent to say how $i = \sqrt{-1}$ acts on TM. This is encoded by and endomorphism $J \in \Gamma(End(TM))$ such that $J^2 = -I$ (called an almos complex structure)

Def A complex manifold structure on M is given by an atlas $M = \bigcup_{\alpha \in I} U_\alpha$ with charts $\varphi_\alpha : U_\alpha \hookrightarrow \mathbb{C}^n$

such that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \underset{\mathbb{C}^n}{\underset{\cap}{\varphi_\alpha(U_\alpha \cap U_\beta)}} \to \underset{\mathbb{C}^n}{\underset{\cap}{\varphi_\beta(U_\alpha \cap U_\beta)}}$

are holomorphic, so the differential is complex-linear.

<u>Observations</u>

(i) It is clear that a complex manifold is an almost complex manifold (structure is preserved by coordinate changes)

(ii) A complex manifold of dimension n is also a real manifold of dimension $2n$.

(iii) A complex manifold has a canonical orientation:
let $e_1, e_2, \dots, e_n$ be a $\mathbb{C}$-basis for T_pM
then $e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n$ is an oriented $\mathbb{R}$-basis for T_pM

(iv) Similarly, the oriented intersection of two transverse complex submanifolds in a complex manifold is always positive.

<u>Examples</u>: (i) Affine space $\mathbb{C}^n$

(ii) Complex projective space
$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^\times \qquad (\mathbb{C}^\times = \mathbb{C} \setminus \{0\}, \text{ multiplicative group})$$
$$= S^{2n+1} / S^1$$
$$= \{\text{complex lines through } 0 \text{ in } \mathbb{C}^{n+1}\}$$

The coordinates $(x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$ become homogeneous coordinates on $\mathbb{CP}^n$: a point in $\mathbb{CP}^n$ is described by an $(n+1)$-tuple $(x_0 : x_1 : \cdots : x_n)$, where x_i are not all zero and we identify $(x_0 : x_1 : \cdots : x_n) = (\lambda x_0 : \cdots : \lambda x_n)$ $\lambda \in \mathbb{C}^\times$

The subset where $x_i \neq 0$ is isomorphic to $\mathbb{C}^n$ via the map
$$\underset{\mathbb{CP}^n}{\underset{\cap}{(x_0 : \cdots : x_n)}} \longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

Since these sets cover $\mathbb{CP}^n$ (and transitions are holomorphic), $\mathbb{CP}^n$ is a <u>complex manifold</u>.

(iii) Any homogeneous polynomial in $x_0, \dots, x_n$ defines a subset of $\mathbb{CP}^n$.

Let $f(x_0, \dots, x_n)$ be a homogeneous polynomial of degree d (e.g. $f = x_0^d + \dots + x_n^d + x_0^2 x_3^{d-2} + \text{etc.}$)

Since $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$, we see that
$$f(\lambda x_0, \dots, \lambda x_n) = 0 \Leftrightarrow f(x_0, \dots, x_n) = 0$$
Thus the set
$$V(f) = \{ (x_0 : \dots : x_n) \in \mathbb{CP}^n \mid f(x_0, \dots, x_n) = 0 \}$$
makes sense. $V(f)$ is called the vanishing locus of f.

If $f, \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}$ never all vanish simultaneously, $V(f)$ is a complex submanifold of $\mathbb{CP}^n$.

We want to compute various topological invariants of $\mathbb{CP}^n$ and $V(f)$
- Chern classes $c_i(M) := c_i(TM) \in H^{2i}(M)$
- Betti numbers $b_i(M) = \dim_{\mathbb{R}} H^i(M; \mathbb{R})$
- Euler characteristic $\chi(M) = \sum_{i=1}^{\dim M} (-1)^i b_i$

We will not use the definition of c_i in terms of curvature, but rather some axioms for Chern classes:

<u>Axiom 1</u>: $c_i(E) \in H^{2i}(M, \mathbb{Z})$ can put $\mathbb{R}$ here and use de Rham.
$c_0(E) = 1$ and $c_i(E) = 0$ for $i > \operatorname{rank}_{\mathbb{C}} E$

<u>Axiom 2</u>: If
$$\begin{array}{ccc} E & \rightarrow & F \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$
is a pull-back: $f^*F \cong E$,

then $f^* c_i(F) = c_i(E)$ (i.e. $c_i(f^*F) = f^* c_i(F)$)

Axiom 3 $\quad c(E \oplus F) = c(E)\,c(F)$

i.e. $\quad c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) \cdot c_{k-i}(F)$

Axiom 4: $\quad c_1(L) = -h \quad$ where $L \to \mathbb{CP}^1 \simeq S^2$ is the tautological line bundle, and $h \in H^2(\mathbb{CP}^2)$ is the class such that $\int_{\mathbb{CP}^2} h = 1$

For any projective space $\mathbb{CP}^n$, there is a tautological line bundle

$$\begin{array}{c} L = \{ (x, \ell) \mid \ell \text{ a line in } \mathbb{C}^{n+1},\ x \in \ell \} \\ \downarrow \\ \mathbb{CP}^n = \{ \ell \mid \ell \text{ a line in } \mathbb{C}^{n+1} \} \end{array}$$

whose fiber at a point is the line represented by that point.

All 4 axioms can be checked from the definition via curvature. The actually characterize the chern classes uniquely.

For today's calculations we will also need the facts

(I) The cohomology of $\mathbb{CP}^n$ is

$$H^*(\mathbb{CP}^n; \mathbb{R}) = \mathbb{R}[h] / (h^{n+1}) = \langle 1, h, h^2, \ldots, h^n \rangle$$

where $h \in H^2(\mathbb{CP}^n; \mathbb{R})$ satisfies $\int_{\mathbb{CP}^1} h = 1$

[so $h^i \in H^{2i}(\mathbb{CP}^n; \mathbb{R})$ is a basis, and $H^{2i+1}(\mathbb{CP}^n, \mathbb{R}) = 0$.]

[$\mathbb{CP}^1$ sits inside $\mathbb{CP}^n$ as the set of points $(x_0 : x_1 : 0 : \cdots : 0)$]

(II) Gauss-Bonnet-Chern theorem. If M is a compact almost complex manifold of complex dimension n, then

$$\int_M c_n(TM) = \chi(M)$$

(III) Letschetz hyperplane theorem: $V(f) \subset \mathbb{CP}^n$

($V(f)$ has complex dimension $n-1$ and real dimension $2n-2$.)

The Betti numbers of $V(f)$ are the same as those of $\mathbb{CP}^n$ below the middle dimension:

$$b_i(V(f)) = b_i(\mathbb{CP}^n) \quad \text{for } i < n-1$$

(IV) for line bundles L_0, L_1 over M, $c_1(L_0 \otimes L_1) = c_1(L_0) + c_1(L_1)$

Let's calculate! (A) let $L \to \mathbb{CP}^n$ be the tautological line bundle

let $f: \mathbb{CP}^1 \to \mathbb{CP}^n$ be the inclusion

$(x_0 : x_1) \mapsto (x_0 : x_1 : 0 : \cdots : 0)$

By (I) $c_1(L) = ah$ for some $a \in \mathbb{R}$

by axiom 2: $a f^*h = f^*(ah) = f^* c_1(L) = c_1(f^*L)$

but f^*L is isomorphic to the tautological bundle on $\mathbb{CP}^1$

so axiom 4 implies $a f^* h$ integrates to -1 on $\mathbb{CP}^1$

$$-1 = \int_{\mathbb{CP}^1} a f^*h = a \int_{\mathbb{CP}^1} f^*h = a \qquad \text{so } a = -1$$

Thus $c_1(L) = -h$ (for any projective space)

(B) Chern classes of $T\mathbb{CP}^n$:

L tautological line bundle is a subbundle of $\underline{\mathbb{C}}^{n+1}$ the trivial bundle. let $L^\perp \subset \underline{\mathbb{C}}^{n+1}$ be the orthogonal complement (w.r.t. hermitian metric)

Fact: $T\mathbb{CP}^n \simeq \mathcal{H}om(L, L^\perp)$

Now $\mathcal{H}om(L, L) \simeq \underline{\mathbb{C}}$ is trivial, so we add it to both sides.

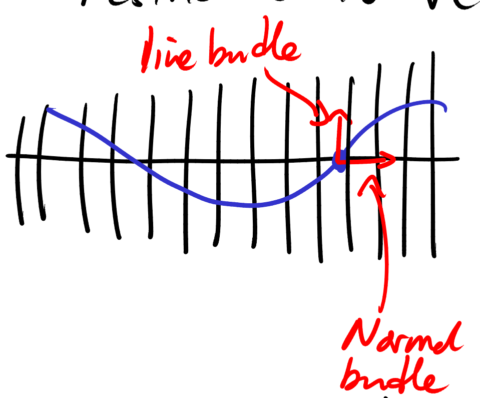
$$\begin{aligned} T\mathbb{CP}^n \oplus \underline{\mathbb{C}} &= \mathcal{H}om(L, L^\perp) \oplus \mathcal{H}om(L, L) \\ &= \mathcal{H}om(L, L^\perp \oplus L) = \mathcal{H}om(L, \underline{\mathbb{C}}^{n+1}) \\ &= L^\vee \otimes \mathbb{C}^{n+1} = L^\vee \oplus \cdots \oplus L^\vee \quad (n \text{ times}) \end{aligned}$$

Since $L \otimes L^\vee \cong \mathcal{H}om(L,L) \cong \underline{\mathbb{C}}$, IV implies
$c_1(L^\vee) = -c_1(L) = h$.

$$\text{so}\quad c(T\mathbb{CP}^n \oplus \underline{\mathbb{C}}) = c(\overbrace{L^\vee \oplus \cdots \oplus L^\vee}^{n+1})$$
$$c(T\mathbb{CP}^n)\, c(\underline{\mathbb{C}}) = c(L^\vee)^{n+1}$$
$$c(T\mathbb{CP}^n) = (1+h)^{n+1}$$
$$c_i(T\mathbb{CP}^n) = \binom{n+1}{i} h^i \quad \text{for} \quad i = 1, 2, \ldots, n$$

in particular $c_n(T\mathbb{CP}^n) = \binom{n+1}{n} h^n = (n+1)h^n$ and $\int_{\mathbb{CP}^n} c_n = n+1 = \chi(\mathbb{CP}^n)$
as expected by (II).

(C) $V(f)$: f a homogeneous polynomial of degree d
$\Longrightarrow$ f is a section of $(L^\vee)^{\otimes d}$
$V(f)$ is the intersection of f with the zero section.
the vertical component of the derivative of f identifies the normal bundle to $V(f)$ with the line bundle restricted to $V(f)$



Let $i : V(f) \to \mathbb{CP}^n$ be the inclusion

We have $i^* T\mathbb{CP}^n \cong TV(f) \oplus i^*(L^\vee)^{\otimes d}$

$$\text{So}\quad c(i^* T\mathbb{CP}^n) = c(TV(f))\, c(i^*(L^\vee)^{\otimes d})$$
$$i^*(1+h)^{n+1} = c(TV(f))\, i^*(1+d\cdot h)$$
$$(1+i^*h)^{n+1} = c(TV(f))\,(1+d\,(i^*h))$$

Note also that $\displaystyle\int_{V(f)} h^{n-1} = d$

More specific example: let $V(f)$ be 1-dimensional in $\mathbb{CP}^2$
What is the genus of $V(f)$ as a surface, in terms of d?

$$(1+h)^{2+1} = c(TV(f))(1+dh)$$
$$1+3h = c(TV(f))(1+dh) \qquad \boxed{\text{since } h^2=0 \text{ on } V(f)}$$
$$c(TV(f)) = (1-dh)(1+3h)$$
$$= 1+(3-d)h$$

So $c_1(TV(f)) = (3-d)h$

$$\chi(V(f)) = \int_{V(f)} (3-d)h = (3-d)\cdot d$$

But $\chi(V(f)) = 2-2g$ so $2-2g = (3-d)\cdot d$

$$g = \frac{(d-1)(d-2)}{2}$$

d	1	2	3	4	5	6
g	0	0	1	3	6	10