Lecture 22

Characteristic Classes and Topology.

Last time we introduced the Chern classes of a complex vector bundle and the Pontryagin classes of a real vector bundle.

Chern class $E \to M$ complex vector bundle of rank r

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$$

$$c_i(E) \in H^{2i}(M, \mathbb{C})$$

Pontryagin class $E \to M$ real vector bundle of rank r

$$p(E) = 1 + p_1(E) + \cdots + p_{\lfloor r/2 \rfloor}(E)$$

$$p_i(E) \in H^{4i}(M, \mathbb{R})$$

Rmk: We actually have $c_i(E) \in H^{2i}(M, \mathbb{Z})$ (E complex)

$p_i(E) \in H^{4i}(M, \mathbb{Z})$ (E real)

There is a relationship between p and c:
If $E \to M$ is a real vector bundle, then $E \otimes_{\mathbb{R}} \mathbb{C} \to M$ is a complex vector bundle whose complex rank = the real rank of E. (think $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^n$)
$E \otimes \mathbb{C}$ is called the complexification of E.

Prop: $c_{2i+1}(E \otimes \mathbb{C}) = 0$ (odd chern classes vanish)

$$c_{2i}(E \otimes \mathbb{C}) = (-1)^i p_i(E)$$

So the Pontryagin classes are essentially the Chern classes of the complexification. We can prove this in a minute.

In the presence of metrics some things simplify:

$\underline{\mathbb{K}=\mathbb{R}}$: Every real vector bundle E admits a metric g, and a metric connection ∇.:

$$d[g(s_1,s_2)] = g(\nabla s_1, s_2) + g(s_1, \nabla s_2) \in \Omega^1(M,\mathbb{R})$$

With respect to an orthonormal frame $s_1,\dots,s_r$: $g(s_i,s_j)=\delta_{ij}$ we get

$$\begin{aligned} 0 = d[\delta_{ij}] &= g(\nabla s_i, s_j) + g(s_i, \nabla s_j) \\ &= g\Big(\sum_k A_i^k s_k, s_j\Big) + g\Big(s_i, \sum_\ell A_j^\ell s_\ell\Big) \\ &= A_i^k \delta_{kj} + A_j^\ell \delta_{i\ell} = A_i^j + A_j^i \end{aligned}$$

So $A_i^j = -A_j^i$ or $\underline{{}^tA = -A \text{ or } A \text{ is skew-symmetric.}}$

The connection matrices F will also be skew-symmetric w.r.t. an orthonormal frame: ${}^tF = -F$

Let X be a skew symmetric matrix ${}^tX = -X$. we have

$$\det(I+X) = \det({}^t(I+X)) = \det(I+{}^tX) = \det(I-X)$$

Thus $\det(I+X)^2 = \det(I+X)\det(I-X) = \det(I-X^2)$

Thus $p(E,\nabla) = \det\Big(\big(I - \big(\tfrac{F}{2\pi}\big)^2\big)^{1/2}\Big) = \det\big(I + \tfrac{F}{2\pi}\big) = \det\big(I - \tfrac{F}{2\pi}\big)$

We could have used any of these expressions to define the Pontryagin <u>class</u>. (Even though the <u>forms</u> are different unless ∇ is a metric connection.)

$\underline{\mathbb{K}=\mathbb{C}}$: The appropriate metric here is a hermitian metric like $\vec{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$ $\vec{w}=(w_1,\dots,w_n)\in\mathbb{C}^n$

$$\langle \vec{z},\vec{w}\rangle = \sum_{i=1}^n \bar{z}_i w_i$$

$h: E\times E\to\mathbb{C}$ $\qquad h(\lambda s_1,s_2)=\bar{\lambda}\, h(s_1,s_2) \qquad \lambda\in\mathbb{C}$

$\mathbb{R}$-bilinear $\qquad h(s_1,\lambda s_2)=\lambda\, h(s_1,s_2)$

$\qquad h(x,x)>0$ if $x\neq 0$.

Every complex vector bundle admits a hermitian metric and a metric connection: $d[h(s_1,s_2)] = h(\nabla s_1,s_2)+h(s_1,\nabla s_2)$
Now if $s_1,\dots,s_r$ is a $\mathbb{C}$-frame which is h-orthonormal:

$$0 = h\Big(\sum_k A_i^k s_k, s_j\Big) + h\Big(s_i, \sum_\ell A_j^\ell s_\ell\Big)$$

$$= \bar{A}_i^j + A_j^i \quad\text{or}\quad {}^t\bar{A}=-A \quad\text{or}\quad A^*=-A \quad\text{or}\quad A^\dagger=-A$$

So A is <u>skew-hermitian.</u>

and F is skew-hermitian as well. This justifies

$$\overline{c(E,\nabla)} = \overline{\det\left(I+\frac{iF}{2\pi}\right)} = \det\left(\left(I+\frac{iF}{2\pi}\right)^*\right) = \det\left(I+\frac{(iF)^*}{2\pi}\right)$$

$$= \det\left(I+\frac{(-i)(-F)}{2\pi}\right) = \det\left(I+\frac{iF}{2\pi}\right) = c(E,\nabla)$$

Thus $c(E)=[c(E,\nabla)]$ is always a <u>real</u> cohomology class.
(Even though the <u>form</u> may not be real unless ∇ is a hermitian connection.)

<u>Proof of proposition:</u> E real v.b., with metric and metric connection ∇
metric on E $\to$ hermitian metric on $E\otimes\mathbb{C}$.
metric conn ∇ $\to$ hermitian conn $\nabla\otimes\mathbb{C}$

The curvature matrix F is essentially the same for both and $\bar{F} = F$, ${}^tF = -F$ ($\Rightarrow F^* = -F$)

Then $$p(E,\nabla) = \det\left(I + \frac{F}{2\pi}\right) = \det\left(I - \frac{F}{2\pi}\right)$$

$$c(E\otimes\mathbb{C}, \nabla\otimes\mathbb{C}) = \det\left(I + \frac{iF}{2\pi}\right) = \det\left(I - \frac{iF}{2\pi}\right)$$

Changing F to $-F$ affects the degree $2i$ part by $(-1)^i$

So $c_i(E\otimes\mathbb{C}) = (-1)^i c_i(E\otimes\mathbb{C})$

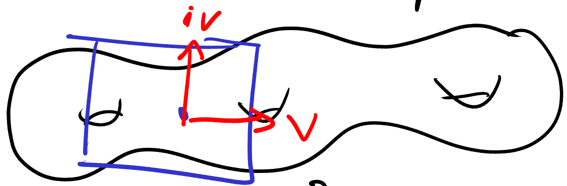
$\Rightarrow c_i(E\otimes\mathbb{C}) = 0$ for i odd.

The degree $4i$ part of $\det\left(I + \frac{iF}{2\pi}\right)$ is $(i)^{2i} = (-1)^i$ times the degree $4i$ part of $\det\left(I + \frac{F}{2\pi}\right)$

So $c_{2i}(E\otimes\mathbb{C}) = (-1)^i p_i(E)$. $\blacksquare$

<u>Connections to topology</u>: Characteristic Classes carry information about the topology of the vector bundle. When the vector bundle is the tangent bundle TM, the characteristic classes carry information about the manifold M itself. There is a whole class of theorems that relate characteristic classes to other seemingly very different topological invariants. We will state some of these theorems now to help us get excited.

Let S be a compact oriented surface w/o boundary.



genus 3.

Pick a metric g on S. The tangent bundle of S turns out to be a complex vector bundle: let $p \in S$: Choose orientation preserving isomorphism $T_pS \to \mathbb{R}^2$. Let $i = \sqrt{-1}$ act by counterclockwise 90° rotation. (makes sense b/c we chose a metric and orientation) Let $c_1(TS)$ be the first chern class of this complex line bundle.

Theorem (Gauss-Bonnet)

$$\langle c_1(TS), [S] \rangle = \int_S c_1(TS, \nabla) = \chi(S) = 2-2g$$

In terms of the Gaussian curvature of g this says

$$\frac{1}{2\pi}\int_S K\, dA = \chi(S)$$

Let M^{4k} be an oriented manifold whose dimension is divisible by 4.

Let $[M] \in H_{4k}(M, \mathbb{Z})$ be the fundamental class associated to the orientation.

On the middle cohomology $H^{2k}(M, \mathbb{R})$ we have a pairing

$$\begin{aligned} H^{2k}(M,\mathbb{R}) \times H^{2k}(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\alpha \quad , \quad \beta) &\longmapsto \langle \alpha \cup \beta, [M] \rangle \end{aligned}$$

or thinking of α and β as differential forms:

$$(\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta$$

This bilinear form is symmetric since $\frac{4k}{2} = 2k$ is even.
A symmetric bilinear form on a real vector space can be diagonalized, and is equivalent to one of the form

$$\begin{bmatrix} \underbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}_{r} & 0 \\ 0 & \underbrace{\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}}_{s} \end{bmatrix}$$

with r positive directions and s negative directions.

The signature is $r - s$.

Let $\sigma(M)$ be the signature of the pairing $(\alpha,\beta) \mapsto \int_M \alpha \wedge \beta$
It is called the signature of the oriented manifold M.

Theorem (Hirzebruch signature theorem) For each k, there is a rational combination of the pontryagin classes L_k of total degree $4k$ such that

$$\sigma(M) = \int_M L_k(TM)$$

for any oriented (compact w/o boundary) $4k$-dimensional manifold M.

We can say what L is: just as $p(E) = \left[\det\left(\left(I - \left(\frac{F}{2\pi}\right)^2\right)^{1/2}\right)\right]$

$$L(E) = \left[\det\left(\left(\frac{\frac{iF}{2\pi}}{\tanh\left(\frac{iF}{2\pi}\right)}\right)^{1/2}\right)\right]$$

And $L_k(E)$ is the degree $4k$-piece of this.
Here $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ is the hyperbolic tangent, regarded as a power series starting with $x \ldots$.