Complex vector bundles; Chern classes, pontryagin classes

We have so far only considered real vector bundles. The basic definitions and the local theory works analogously for Complex vector bundles.

Def A complex vector bundle $\pi : E \to M$ of rank r is a smooth manifold E and a smooth map $\pi : E \to M$ such that each fiber $E_x = \pi^{-1}(x)$ is a complex vector space, and there are local trivializations

$\forall x \in M \ \exists\ U \ni x$ open $\exists\ \phi : \pi^{-1}(U) \to U \times \mathbb{C}^r$ s.t.

(1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U \times \mathbb{C}^r \arrow[d, "pr_1"] \\ U \arrow[r, "id"] & U \end{tikzcd}$$
commutes

(2) $\phi|_{\pi^{-1}(x)} : E_x \to \{x\} \times \mathbb{C}^r$ is an isomorphism of complex vector spaces.

- Transition functions are maps $\varphi_{12} : U_1 \cap U_2 \to GL(r, \mathbb{C})$
- Sections $\Gamma(E)$ is a complex vector space, and a module over complex-valued functions $C^\infty(M, \mathbb{C})$
- connections are required to be $\mathbb{C}$-linear: $\nabla(as) = a\nabla(s)$, $a \in \mathbb{C}$
- local frame $\{s_i\}_{i=1}^r$ is a collection of sections that forms a $\mathbb{C}$-basis of each fiber.
- w.r.t. a frame, the connection has the form $\nabla = d + A$ for a 1-form with values in $r \times r$ complex matrices, F is similarly a 2-form with values in $r \times r$ complex matrices.
- More invariantly F_∇ is a section of $\Lambda^2 T^*M \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{C}}(E)$
- Multilinear algebra is done over $\mathbb{C}$: $E^\vee = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$

$\mathrm{End}_{\mathbb{C}}(E) = E^\vee \otimes_{\mathbb{C}} E \qquad tr : E^\vee \otimes_{\mathbb{C}} E \to \mathbb{C}$, etc.

Now let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be the ground field, we consider $\mathbb{K}$-vector bundles and $\mathbb{K}$-linear connections.
$[\ E^\vee = \mathrm{Hom}_{\mathbb{K}}(E,\mathbb{K}),\quad \mathcal{E}nd(E) = \mathcal{E}nd_{\mathbb{K}}(E),\quad \otimes = \otimes_{\mathbb{K}},$ etc. $]$

Recall: ∇ on E induces connections on $E^\vee$, $\mathcal{E}nd(E)$, etc, which are compatible with contractions and satisfy Leibniz rule w.r.t. $\otimes$

$$\Omega^p(M,E) = \Gamma(\Lambda^p T^*M \otimes_{\mathbb{R}} E) \qquad d_\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E)$$

$\Omega^p(M, \mathcal{E}nd(E))$ has $d_\nabla = d_{\nabla^{\mathcal{E}nd(E)}}$

$F_\nabla \in \Omega^2(M, \mathcal{E}nd(E))$ satisfies $d_\nabla F_\nabla = 0$

Now observe that $\Omega^\bullet(M, \mathcal{E}nd(E))$ is actually a ring

Consider $\omega \otimes A$ and $\eta \otimes B$, $\omega, \eta \in \Omega^\bullet(M)$ $A, B \in \Gamma(\mathcal{E}nd(E))$

Product $(\omega \otimes A)(\eta \otimes B) = (\omega \wedge \eta) \otimes (A \circ B)$

<u>Lemma</u>: d_∇ is a derivation of this product.

<u>Proof</u> For 0-forms this follows from compatibility with contractions

$\circ : \mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \to \mathcal{E}nd(E)$ corresponds to

$$C : E^\vee \otimes E \otimes E^\vee \otimes E \to E^\vee \otimes E$$

contract these

So for $A, B \in \Gamma(\mathcal{E}nd(E))$

$$\nabla(A \circ B) = \nabla(C(A \otimes B)) = C\, \nabla(A \otimes B) = C(\nabla A \otimes B + A \otimes \nabla B)$$
$$= (\nabla A) \circ B + A \circ (\nabla B)$$

$$d_\nabla(\omega \wedge \eta \otimes AB) = d(\omega \wedge \eta) \otimes AB + (-1)^{p+q} (\omega \wedge \eta) \wedge \nabla AB$$

$$= (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta)\, AB + (-1)^{p+q} (\omega \wedge \eta) \wedge ((\nabla A) B + A \nabla B)$$

$$= (d\omega \otimes A)(\eta \otimes B) + (-1)^p (\omega \otimes A)(d\eta \otimes B)$$

$$+ (-1)^p (\omega \wedge \nabla A)(\eta \otimes B) + (-1)^{p+q} (\omega \otimes A)(\eta \wedge \nabla B)$$

$$= \left(d\omega \otimes A + (-1)^p \omega \wedge \nabla A\right)(\eta \otimes B) + (-1)^p (\omega \otimes A)\left(d\eta \otimes B + (-1)^q (\eta \wedge \nabla B)\right)$$
$$= \quad d_\nabla(\omega \otimes A)\cdot(\eta \otimes B) + (-1)^p (\omega \otimes A)\, d_\nabla(\eta \otimes B) \qquad \checkmark$$

Now we can consider expressions like $F_\nabla^k = \underbrace{F_\nabla \cdot F_\nabla \cdot \cdots \cdot F_\nabla}_{k} \in \Omega^{2k}(M, End(E))$

<u>Corollary of Bianchi identity</u>: $d_\nabla(F_\nabla^k) = 0$

<u>Proof</u>: $d_\nabla(F_\nabla^k) = (d_\nabla F_\nabla)\cdot F_\nabla^{k-1} + F_\nabla\, d_\nabla(F_\nabla^{k-1})$

Use $d_\nabla F_\nabla = 0$ and induction on k.

<u>Corollary</u>: $tr(F_\nabla^k) \in \Omega^{2k}(M, \mathbb{K})$ is closed.

we saw before that $d\,tr = tr\, d_\nabla$, so

$d\,tr(F_\nabla^k) = tr(d_\nabla(F_\nabla^k)) = 0.$

We also want to show that $[tr(F_\nabla^k)] \in H^{2k}(M, \mathbb{K})$ does not depend on the choice of connection.

<u>Lemma</u>: let $\alpha_t \in \Omega^p(M)$ $t \in [0,1]$ be a smooth path of closed forms. Suppose there is a smooth path $\beta_t \in \Omega^{p-1}(M)$ $t \in [0,1]$ such that

$$\frac{\partial \alpha_t}{\partial t} = d(\beta_t) \qquad \text{(The time derivative is exact.)}$$

Then $\alpha_1 - \alpha_0 = d\int_0^1 \beta_t\, dt$, and hence $[\alpha_1] = [\alpha_0] \in H^p(M)$

Now let ∇_t $t \in [0,1]$ be a smooth family of connections

<u>NB</u>: Any two connections ∇_0 and ∇_1 can be connected by a path of the form $\nabla_t = \nabla_0 + ta$ where $a = \nabla_1 - \nabla_0 \in \Omega^1(M, End(E))$

Let $\quad a_t = \frac{\partial \nabla_t}{\partial t} = \lim_{h\to 0} \frac{\nabla_{t+h} - \nabla_t}{h} \in \Omega^1(M, End(E))$

Let $F_{\nabla_t} = (d_{\nabla_t})^2$ be the curvature.

Now we need $\frac{\partial F_{\nabla_t}}{\partial t} = \lim_{h\to 0} \frac{F_{\nabla_{t+h}} - F_{\nabla_t}}{h}$

$$\begin{aligned} F_{\nabla_{t+h}} - F_{\nabla_t} &= d_{\nabla_t}(\nabla_{t+h} - \nabla_t) + (\nabla_{t+h} - \nabla_t)^2 \\ &= d_{\nabla_t}(h a_t + o(h)) + (h a_t + o(h))^2 \\ &= h\, d_{\nabla_t} a_t + o(h) \end{aligned}$$

$$\therefore \frac{\partial F_{\nabla_t}}{\partial t} = d_{\nabla_t} a_t = d_{\nabla_t}\left(\frac{\partial \nabla_t}{\partial t}\right).$$

Now what about $F_{\nabla_t}^k$?

$$\begin{aligned} \frac{\partial}{\partial t}\left(F_{\nabla_t}^k\right) &= \sum_{i=0}^{k-1} F_{\nabla_t}^i \frac{\partial F_{\nabla_t}}{\partial t} F_{\nabla_t}^{k-1-i} \\ &= \sum_{i=0}^{k-1} F_{\nabla_t}^i \left(d_{\nabla_t} a_t\right) F_{\nabla_t}^{k-1-i} \\ &= \sum_{i=0}^{k-1} d_{\nabla_t}\left(F_{\nabla_t}^i a_t F_{\nabla_t}^{k-1-i}\right) \quad \text{by Bianchi } d_{\nabla_t} F_{\nabla_t} = 0 \\ &= d_{\nabla_t}\left(\sum_{i=1}^{k-1} F_{\nabla_t}^i a_t F_{\nabla_t}^{k-1-i}\right) \end{aligned}$$

Now consider $tr(F_{\nabla_t}^k)$

$$\frac{\partial}{\partial t}\left(tr(F_{\nabla_t}^k)\right) = tr\left(\frac{\partial}{\partial t} F_{\nabla_t}^k\right) = tr\left(d_{\nabla_t}\left(\sum_{i=0}^{k-1} F_{\nabla_t}^i a_t F_{\nabla_t}^{k-1-i}\right)\right)$$
$$= d\, tr\left(\sum_{i=0}^{k-1} F_{\nabla_t}^i a_t F_{\nabla_t}^{k-1-i}\right)$$

Thus $\frac{\partial}{\partial t}\left(tr(F_{\nabla_t}^k)\right)$ is exact! and

$$tr(F_{\nabla_1}^k) - tr(F_{\nabla_0}^k) = d\int_0^1 tr\left(\sum_{i=0}^{k-1} F_{\nabla_t}^i a_t F^{k-1-i}\right) dt$$

This is called the Chern-Simons form for the invariant polynomial $tr(X^k)$ and the path ∇_t.
It is not always closed, but in some cases it is, for instance if ∇_0 and ∇_1 are flat, or if $2k > \dim M$.
In the latter case, we only get something interesting if $\dim M = 2k-1$

<u>Conclusion:</u> $[tr(F_\nabla^k)] \in H^{2k}(M, \mathbb{K})$ is a well-defined cohomology class depending only on the vector bundle E, not on the connection.

- Certain polynomial combinations of them are the chern classes, pontryagin classes, etc.
- We can even use power series, since $F_\nabla^k = 0$ as soon as $2k > \dim M$

Traditional characteristic classes

$\mathbb{K} = \mathbb{C}$:

Consider $\exp(X) = 1 + X + \frac{X^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{X^k}{k!}$

and define $ch(E,\nabla) = tr\left(\exp\left(\frac{\sqrt{-1}}{2\pi} F_\nabla\right)\right)$

$$= tr\left(I + \left(\frac{\sqrt{-1}}{2\pi} F_\nabla\right) + \frac{\left(\frac{\sqrt{-1}}{2\pi} F_\nabla\right)^2}{2} + \cdots\right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{\sqrt{-1}}{2\pi}\right)^k \frac{1}{k!} \, tr\left(F_\nabla^k\right) \in \Omega^{even}(M, \mathbb{C})$$

This is a finite sum of forms of different even degrees.

$ch(E,\nabla)$ is called the Chern Character form and the class $ch(E) = [ch(E,\nabla)] \in H^{even}(M, \mathbb{C})$ is called the Chern Character.

$ch_k(E) = \left(\frac{\sqrt{-1}}{2\pi}\right)^k \frac{1}{k!} \left[tr\left(F_\nabla^k\right)\right] \in H^{2k}(M, \mathbb{C})$ is the degree $2k$ piece of the chern character.

Rmk $ch(E)$ is actually a rational cohomology class, but in this approach that is not clear.

Now consider $c(E,\nabla) = \det\left(I + \frac{\sqrt{-1}}{2\pi} F_\nabla\right) \in \Omega^{even}(M, \mathbb{C})$

$c(E) = [c(E,\nabla)] \in H^{even}(M, \mathbb{C})$

These are the total Chern form and total Chern class. The degree $2k$ piece $c_k(E)$ is the k-th chern class.

In fact $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$
where r is the rank of E.

To fit $c(E,\nabla)$ into the framework developed so far, we can use the identity $\det(\exp(X)) = \exp(\mathrm{tr}(X))$ for any matrix X, which implies

$$\det(I+X) = \exp(\mathrm{tr}(\log(I+X)))$$

Provided $\log(I+X)$ exists.
We also have a power series

$$\log(I+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}$$

Thus
$$c(E,\nabla) = \exp\left(\mathrm{tr}\left(\log\left(I + \tfrac{\sqrt{-1}}{2\pi} F_\nabla\right)\right)\right)$$
$$= \exp\left(\mathrm{tr}\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\tfrac{\sqrt{-1}}{2\pi}\right)^k F_\nabla^k\right)\right)$$

All the power series are actually finite sums since there are no forms of degree $>\dim M$.

$\underline{\mathbb{K} = \mathbb{R}}$:

$$p(E,\nabla) = \det\left(\left(I - \left(\tfrac{1}{2\pi} F_\nabla\right)^2\right)^{1/2}\right) \quad \text{Pontryagin Form}$$

$$p(E) = [p(E,\nabla)] \quad \text{Total Pontryagin Class.}$$

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots + p_{\lfloor r/2 \rfloor}(E)$$
$$p_i(E) \in H^{4i}(M;\mathbb{R}) \quad \text{Pontryagin Classes.}$$