Lecture 20

Bianchi Identity; Induced connections; Exterior covariant derivative

Recall: $\pi: E \to M$ vector bundle, ∇ connection, $\underset{\text{curvature}}{F} \in \Omega^2(M, \mathrm{End}(E))$

With respect to a local frame $\{s_i\}_{i=1}^r$, $\nabla: \Gamma(E) \to \Omega^1(E)$ has the form

$$\nabla = d + A$$

where $A = \sum_{\mu=1}^n A_\mu dx^\mu$ is a matrix-valued 1-form, and the curvature F

is represented by a matrix-valued 2-form. They are related by the structure equation:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$F = dA + \underbrace{[A,A]}_{[A,A](X,Y) = [A(X), A(Y)]} = dA + \underbrace{A \wedge A}_{\text{combine wedge and matrix mult.}}$$

New: there is also an equation relating dF to F and A

Thm (2nd or differential) Bianchi identity: With respect to the frame $\{s_i\}$

$$dF = \underbrace{F \wedge A}_{} - \underbrace{A \wedge F}_{}$$

combine wedge and matrix multiplication

Proof: Homework. (Hint: use structure equation)

We can reformulate things if we extend the notion of covariant differentiation just a bit:

Induced connections: Given a connection in $E \to M$, we can define connections in all bundles of tensors "made from E" such as $E^\vee$, $\mathrm{End}(E)$, $E \otimes E$, and so on.
The definition follows from two axioms
(1) Leibniz rule : $\nabla(\alpha \otimes \beta) = (\nabla\alpha)\otimes\beta + \alpha \otimes (\nabla\beta)$
(2) ∇ is compatible with contractions.
(Natural transformations are covariant constant).

Let us see this in action: On $E^\vee$ and $E^\vee \otimes E$ there are induced connections $\nabla^\vee$ and $\nabla^{E^\vee \otimes E}$. Local sections of $E^\vee \otimes E$ are generated by $\lambda \otimes v$, where $\lambda \in \Gamma(E^\vee)$ and $v \in \Gamma(E)$
By (1), we must have

$$\nabla^{E^\vee \otimes E}(\lambda \otimes v) = (\nabla^\vee \lambda)\otimes v + \lambda \otimes (\nabla v)$$

Also, there is a contraction

$$C : E^\vee \otimes E \longrightarrow \mathbb{R}$$
$$\lambda \otimes v \longmapsto \lambda(v)$$

The connection on $\mathbb{R}$ is trivial $\nabla^{\mathbb{R}} = d$, and compatibility with contraction says $\nabla^{\mathbb{R}} \circ C = C \circ \nabla^{E^\vee \otimes E}$ or

$$d(C(\lambda \otimes v)) = C((\nabla^\vee\lambda)\otimes v + \lambda \otimes (\nabla v))$$

$$d[\lambda(v)] = (\nabla^\vee \lambda)(v) + \lambda(\nabla v)$$

so

$$(\nabla^\vee\lambda)(v) = d[\lambda(v)] - \lambda(\nabla v)$$

and this formula completely determines $\nabla^\vee \lambda$ for any $\lambda \in \Gamma(E^\vee)$
With ∇ and $\nabla^\vee$, we get connections on $(E^{\otimes k}) \otimes (E^\vee)^{\otimes l}$, and in particular on $\mathrm{End}(E) \cong E^\vee \otimes E$.

HW: find connection matrix of $\nabla^\vee$ and $\nabla^{\mathrm{End}(E)}$

We can also combine covariant differentiation with the (connection-independent) exterior derivative on forms.
Let $\Omega^p(M,E) = \Gamma(\Lambda^p T^*M \otimes E)$ be the space of p-forms with values in E. Then there is a unique operator

$$d_\nabla : \Omega^p(M,E) \longrightarrow \Omega^{p+1}(M,E) \quad \text{satisfying}$$

(1) (Extends connection): for $s \in \Omega^0(M,E) = \Gamma(E)$,

$$d_\nabla(s) = \nabla s \in \Omega^1(M,E)$$

(2) (Leibniz rule) for $\omega \in \Omega^p(M,E)$ and $s \in \Gamma(E)$

$$d_\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega}\, \omega \wedge (\nabla s)$$

d_∇ is called covariant exterior derivative.

The curvature has a nice interpretation in terms of d_∇. We consider the operator $(d_\nabla)^2 : \Omega^p(M,E) \to \Omega^{p+2}(M,E)$

Let's apply $(d_\nabla)^2$ to a section $s \in \Omega^0(M,E) = \Gamma(E)$

$$d_\nabla(s) = \nabla s \in \Omega^1(M,E)$$

$$\nabla s = \sum_\mu dx^\mu \otimes \nabla_{\partial_\mu} s$$

$$d_\nabla d_\nabla s = d_\nabla(\nabla s) = d_\nabla\Big(\sum_\mu dx^\mu \otimes \nabla_{\partial_\mu} s\Big)$$

$$= \sum_\mu d\,dx^\mu \otimes \nabla_{\partial_\mu} s - dx^\mu \wedge \nabla(\nabla_{\partial_\mu} s)$$

$$= \sum_\mu -dx^\mu \wedge \Big(\sum_\nu dx^\nu (\nabla_{\partial_\nu} \nabla_{\partial_\mu} s)\Big)$$

$$= \sum_{\mu<\nu} \left(\nabla_{\partial_\mu} \nabla_{\partial_\nu} s - \nabla_{\partial_\nu} \nabla_{\partial_\mu} s\right) dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} (F_{\mu\nu} \cdot s)\, dx^\mu \wedge dx^\nu$$

Conclusion $(d_\nabla)^2 s = F_\nabla \cdot s$ where $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$ is the curvature! A similar computation shows that
$$d_\nabla d_\nabla(\omega) = \underbrace{F_\nabla \wedge \omega}_{\text{wedge forms, apply endomorphism to section}} \quad \text{for } \omega \in \Omega^p(M,E)$$

So: $\boxed{(d_\nabla)^2 = F_\nabla}$

We can also reformulate the Bianchi Identity:

Since F_∇ is an $\mathrm{End}(E)$-valued form, it lives in $\Omega^2(M, \mathrm{End}(E))$

But $\mathrm{End}(E)$ has a natural connection, also called ∇, and so there is an exterior covariant derivative
$$d_\nabla : \Omega^p(M, \mathrm{End}(E)) \to \Omega^{p+1}(M, \mathrm{End}(E))$$

<u>Thm</u>: (Bianchi identity) $\quad d_\nabla F_\nabla = 0 \in \Omega^3(M, \mathrm{End}(E))$

<u>Pf</u> HW: (Hint compute $(d_\nabla)^3 s$ two ways)

With respect to local frame, any $\omega \in \Omega^p(M,E)$ becomes just a vector-valued p-form, and
$$d_\nabla(\omega) = d\omega + \underbrace{A \wedge \omega}_{\text{combine matrix action with wedge product.}}$$

HW: Figure out the action of d_∇ on $\Omega^p(M, \mathrm{End}(E))$ which is a bit different. Show that the two versions of the Bianchi identity are equivalent.

Observations: In the Riemannian context, where ∇ = Levi-Civita connection of (M,g), the Bianchi identity holds for the Riemann curvature tensor R: it boils down to

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0$$

where this covariant derivative is on $End(TM)$.

- The E-valued forms $(\Omega^p(M,E), d_\nabla)$ are kind of like the de Rham complex $(\Omega^p(M), d)$, except that d_∇ is not a differential, since $d_\nabla^2 = F_\nabla$ which is not necessarily zero.
  But if the connection ∇ is flat, $F_\nabla = 0$, then $d_\nabla^2 = 0$ and we have a complex!
  The cohomology is defined as usual

$$H^p(M,(E,\nabla)) := \frac{\ker\left(d_\nabla : \Omega^p(M,E) \to \Omega^{p+1}(M,E)\right)}{\operatorname{Im}\left(d_\nabla : \Omega^{p-1}(M,E) \to \Omega^p(M,E)\right)}$$

This is the twisted de Rham cohomology of M with coefficients in the flat vector bundle (E,∇).
Sometimes people write $H^p(M,E)$, but it does depend on ∇.

Example:
- E is trivial bundle $\mathbb{R}\times M$, $\nabla = d$ then $(\Omega^\bullet(M,E), d_\nabla) = (\Omega^\bullet(M), d)$, and we get ordinary de Rham cohomology
- $H^0(M,(E,\nabla)) = \{ s \in \Gamma(E) \mid \nabla s = 0 \}$ is the space of global covariant constant sections, which may very well zero.

Characteristic forms and Classes associated to a vector bundle may be constructed from "invariant polynomial functions" of curvature. Since the curvature F_∇ has values in $End(E)$, we can think of it as matrix-valued, but the matrix is only defined up to conjugation $F \to G^{-1}FG$. So we can get something well defined if we apply a conjugation invariant function to F_∇.

As a warmup, consider the simplest such function, the trace

$$tr : End(E) \longrightarrow \underline{\mathbb{R}} = \text{trivial bundle.}$$

$$tr : \Omega^p(M, End(E)) \longrightarrow \Omega^p(M, \mathbb{R}) \quad = \text{ordinary forms.}$$

Since tr is a contraction, and the connection on $End(E)$ is compatible with contractions, we know

$$d(tr\,\varphi) = tr(\nabla\varphi) \qquad \varphi \in \Gamma(End(E))$$

and so

$$d(tr\,\alpha) = tr(d_\nabla \alpha) \qquad \alpha \in \Omega^p(M, End(E))$$

Now consider $tr(F_\nabla) \in \Omega^2(M)$

we have $d(tr(F_\nabla)) = tr(d_\nabla F_\nabla) = 0$ by Bianchi identity

So $tr(F_\nabla)$ is closed, and defines a class $[tr(F_\nabla)] \in H^2_{dR}(M)$

But moreover, this class does not depend on the choice of connection!

Suppose $\tilde\nabla$ is another connection on E:

$$\tilde\nabla = \nabla + a \quad \text{for some } a \in \Omega^1(M, End(E))$$

$$d_{\tilde\nabla} = d_\nabla + a$$

$$F_{\tilde\nabla} s = d_{\tilde\nabla} d_{\tilde\nabla} s = (d_\nabla + a)(d_\nabla + a)s = (d_\nabla + a)(d_\nabla s + as)$$
$$= d_\nabla d_\nabla s + d_\nabla(as) + a\, d_\nabla s + a \wedge a s$$
$$= F_\nabla s + (d_\nabla a)s - a\, d_\nabla s + a\, d_\nabla s + a \wedge a s$$
$$= (F_\nabla + d_\nabla a + a \wedge a)s$$

So $\mathrm{tr}(F_{\tilde\nabla}) = \mathrm{tr}\, F_\nabla + \mathrm{tr}(d_\nabla a) + \mathrm{tr}(a \wedge a)$
$= \mathrm{tr}\, F_\nabla + d(\mathrm{tr}(a)) + 0$

Thus the difference $\mathrm{tr}(F_{\tilde\nabla}) - \mathrm{tr}(F_\nabla) = d\, \mathrm{tr}(a)$ is exact, and the classes $[\mathrm{tr}(F_{\tilde\nabla})] = [\mathrm{tr}(F_\nabla)] \in H^2_{dR}(M)$.

For real vector bundles, the class $[\mathrm{tr}(F_\nabla)]$ is unfortunately always zero. For complex vector bundles, it is the <u>first chern class</u>. More on the general story next time.