Lecture 2: Changes of trivialization, examples

<u>Recall</u>: Vector bundle $\pi : E \to X$

X, E smooth manifolds, π smooth map.

Fiber $E_x = \pi^{-1}(x)$ has a vector space structure

X is covered by open sets $\{U_\alpha\}_{\alpha \in A}$ such that

a local trivialization ϕ_α exists over U_α:

(1) $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ Diffeomorphism

(2)
$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times \mathbb{R}^k \arrow[d, "pr_1"] \\ U_\alpha \arrow[r, equal, "id"] & U_\alpha \end{tikzcd}$$
commutes

(3) $\phi_\alpha|_{\pi^{-1}(x)} : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^k$ is linear isomorphism.

Note the trivialization is <u>not unique</u>: What is the ambiguity?

Consider $\phi_1 : \pi^{-1}(U) \to U \times \mathbb{R}^k$ $\phi_2 : \pi^{-1}(U) \to U \times \mathbb{R}^k$
two maps satisfying (1), (2), (3)

What about $\phi_2 \circ \phi_1^{-1} : U \times \mathbb{R}^k \longrightarrow U \times \mathbb{R}^k$

(1) $\Rightarrow$ $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism

(2) $\Rightarrow$
$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r, "\phi_2\phi_1^{-1}"] \arrow[d] & U \times \mathbb{R}^k \arrow[d] \\ U \arrow[r, equal] & U \end{tikzcd}$$
commutes

(3) $\Rightarrow$ $\quad \phi_2 \circ \phi_1^{-1} : \{x\} \times \mathbb{R}^k \longrightarrow \{x\} \times \mathbb{R}^k$

is a linear isomorphism for each x.

But what is a linear iso $\mathbb{R}^k \to \mathbb{R}^k$? Multiplication by an invertible matrix:

$$GL(k, \mathbb{R}) = \{ A \ \ k \times k \text{ matrix} \mid \det(A) \neq 0 \}$$

This is a Lie group.

(3) implies $\quad \phi_2 \circ \phi_1^{-1} : \ U \times \mathbb{R}^k \longrightarrow U \times \mathbb{R}^k$

has the form $\quad (x, v) \longmapsto (x, A(x) \cdot v)$

where $A(x) \in GL(k, \mathbb{R})$ for each x.

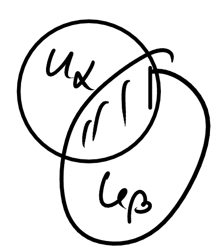
A map of such form is smooth iff the map $\left\{ \begin{array}{l} U \longrightarrow GL(k, \mathbb{R}) \\ x \longmapsto A(x) \end{array} \right\}$ is smooth.

<u>To summarize</u>: The "<u>difference</u>" between two trivializations over the same open set U is encoded by a map $A : U \longrightarrow GL(k, \mathbb{R})$

Return to $\pi : E \to X$, covering $\{U_\alpha\}_{\alpha \in A}$, trivializations ϕ_α

let α and β be indices such that $U_\alpha \cap U_\beta \neq \emptyset$

Then over $U_\alpha \cap U_\beta$ we have two trivializations



$$\phi_\alpha : \pi^{-1}(U_\alpha \cap U_\beta) \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$
$$\phi_\beta : \pi^{-1}(U_\alpha \cap U_\beta) \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

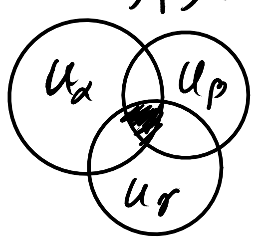
The difference $\phi_\beta \circ \phi_\alpha^{-1}$ has the form $(x,v) \mapsto (x, A_{\alpha\beta}(x)\cdot v)$

So we get a map $A_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{R})$

This $A_{\alpha\beta}$ is the <u>transition matrix</u> <u>from</u> the α-chart <u>to</u> the β-chart.

If we swap α and β, we find $A_{\beta\alpha}(x) = A_{\alpha\beta}(x)^{-1}$

The matrices have another property coming from triple overlaps
let α, β, γ be such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$



over $U_\alpha \cap U_\beta \cap U_\gamma$ we have $\phi_\alpha, \phi_\beta, \phi_\gamma$

The obvious relation
$$(\phi_\gamma \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ \phi_\alpha^{-1}) = \phi_\gamma \circ \phi_\alpha^{-1}$$

implies $\boxed{A_{\beta\gamma}(x) \cdot A_{\alpha\beta}(x) = A_{\alpha\gamma}(x)}$

This is called the <u>cocycle condition</u> for $\{A_{\alpha\beta}(x)\}_{\alpha,\beta \in A}$

In the homework you will show that a vector bundle may be determined by a covering $\{U_\alpha\}_{\alpha\in A}$ of X and a collection $\{A_{\alpha\beta}\}$ satisfying the cocycle condition.

What about sections? $s : X \to E$ becomes in (U_α, ϕ_α) trivialization

$$\begin{array}{ccccc} U_\alpha & \xrightarrow{s} & \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{R}^k \\ x & \longmapsto & & & (x, s_\alpha(x)) \end{array}$$

where $s_\alpha : U_\alpha \to \mathbb{R}^k$ is a vector valued function.

for $x \in U_\alpha \cap U_\beta$, the obvious relation

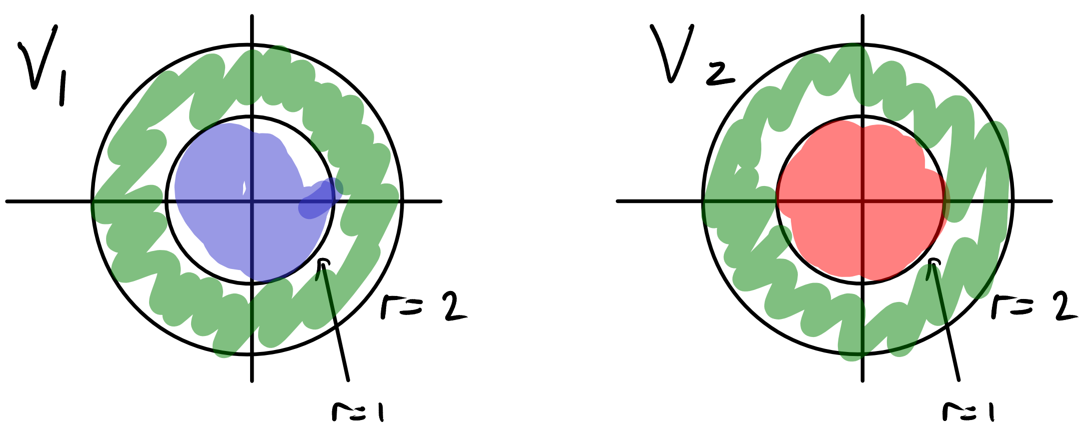
$$(\phi_\beta \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ s) = \phi_\beta \circ s$$

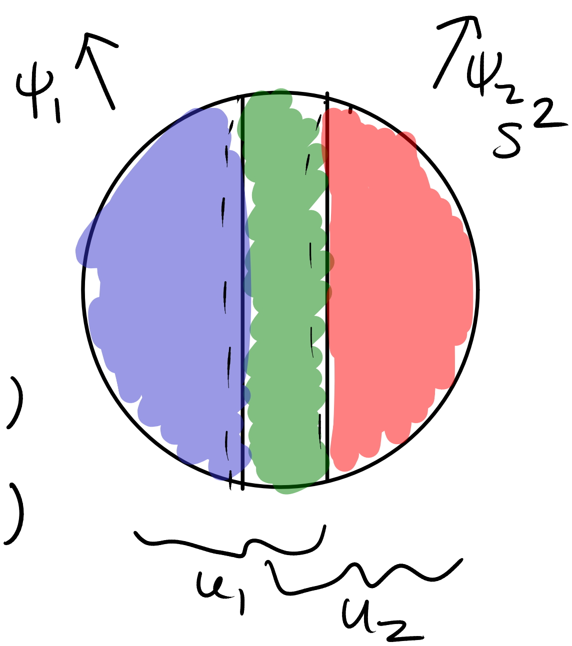
implies $$A_{\alpha\beta}(x) \cdot s_\alpha(x) = s_\beta(x)$$

So: A section is equivalent to a collection $\{s_\alpha\}$ of vector valued functions on charts $\{U_\alpha\}$, subject to the condition that, on overlaps $U_\alpha \cap U_\beta$, they are related by the transition matrices $A_{\alpha\beta}$.

<u>Examples</u> • Tangent bundle of a 2-dimensional sphere S^2

Build an S^2 this way: Take two copies of a disk of radius 2





The gluing map in the green region is $\psi_2 \circ \psi_1^{-1}$:

$$\{1<r<2\} \to \{1<r<2\}$$

$$(r,\theta) \longmapsto (3-r,\theta)$$

(polar coords of course)

The transition matrices for the tangent bundle are the derivative of the gluing maps defining the manifold.

At $x \in U_1 \cap U_2$, we have associated $\varphi_1(x) \in V_1$
$\varphi_2(x) \in V_2$

The gluing map $\varphi_2 \circ \varphi_1^{-1}$ sends $\varphi_1(x) \mapsto x \mapsto \varphi_2(x)$
Since V_1 and V_2 are open sets in $\mathbb{R}^2$, we may regard

$$D(\varphi_2 \circ \varphi_1^{-1})_{\varphi_1(x)} : T_{\varphi_1(x)} V_1 \to T_{\varphi_2(x)} V_2$$

as a plain old matrix. Call it $A(x) \in GL(2,\mathbb{R})$
then
$$\left\{ \begin{array}{ccc} U_1 \cap U_2 & \to & GL(2,\mathbb{R}) \\ x & \longmapsto & A(x) \end{array} \right\}$$
is the transition matrix of the tangent bundle of S^2

Explicity: $F = \varphi_2 \circ \varphi_1^{-1}$ $(r, \theta) \to (3-r, \theta)$
Cartesian coords $x = r\cos\theta$
$y = r \sin\theta$

$$\text{polar}(3-r, \theta) = \text{cartesian } ((3-r)\cos\theta, (3-r)\sin\theta)$$

$$= \text{cartesian}\left((3-\sqrt{x^2+y^2}) \frac{x}{\sqrt{x^2+y^2}}, (3-\sqrt{x^2+y^2}) \frac{y}{\sqrt{x^2+y^2}} \right)$$

$$= \left(\frac{3}{\sqrt{x^2+y^2}} - 1 \right) \cdot (x, y) = f(x,y) \cdot (x,y)$$

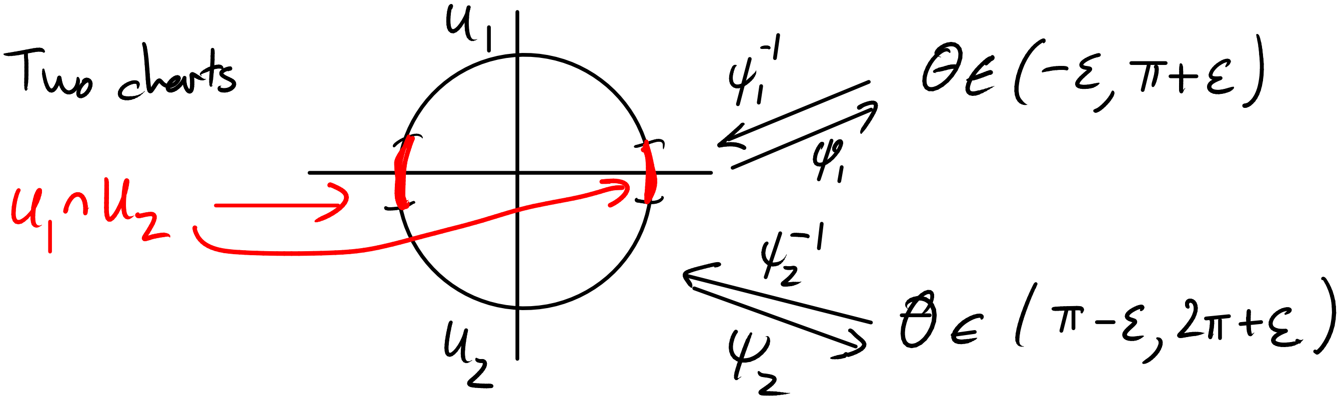
$$A(x,y) = \begin{bmatrix} f + \frac{\partial f}{\partial x} x & \frac{\partial f}{\partial y} x \\ \frac{\partial f}{\partial x} y & f + \frac{\partial f}{\partial y} y \end{bmatrix}$$

where
$$f = \frac{3}{\sqrt{x^2+y^2}} - 1$$

$\underline{k=1}$ $GL(1,\mathbb{R}) = \mathbb{R}\setminus\{0\}$

Möbius bundle over $S^1 = \{(\cos\theta, \sin\theta) \mid \theta \in [0,2\pi)\} \subset \mathbb{R}^2$

Two charts



φ_i^{-1} has the form $\theta \longmapsto (\cos\theta, \sin\theta)$

$U_1 \cap U_2$ has two components: $\underline{\text{Left}}$ and $\underline{\text{Right}}$

on $\underline{\text{left}}$ $\quad \varphi_2 \circ \varphi_1^{-1} : (\pi-\varepsilon, \pi+\varepsilon) \to (\pi-\varepsilon, \pi+\varepsilon)$ is the Identity map.

choose the transition matrix $A = 1 \in GL(1,\mathbb{R})$

on $\underline{\text{Right}}$ $\quad \varphi_2 \circ \varphi_1^{-1} : (-\varepsilon, \varepsilon) \longrightarrow (2\pi-\varepsilon, 2\pi+\varepsilon)$

is the map $x \longmapsto x + 2\pi$

Choose $A = -1 \in GL(1,\mathbb{R})$ over this part.

Möbius band!