Lecture 19

Curvature of Connections in general vector bundles.

Recall $\pi : E \to M$ a vector bundle (could be tangent bundle)
local trivializations of E correspond to local frames:
Over $U \subset M$, we have sections $s_i : U \to \pi^{-1}(U)$ $i = 1, \dots, r$
where r is the rank of E, such that
$\{s_1(p), s_2(p), \dots, s_r(p)\}$ is a basis for $E_p = \pi^{-1}(p)$.

Any section s can be written as $s = \sum_{j=1}^{r} a^j s_j$ for functions $a^j : U \to \mathbb{R}$

Let $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ be a connection, then

$$\nabla_{\frac{\partial}{\partial x^\mu}} s_i = \sum_{j=1}^{r} A_{\mu i}^{\;j} s_j \qquad (A_{\mu i}^{\;j} \text{ are Christoffel symbols})$$

We are using the notation where $\mu = 1, \dots, n$
indexes the coordinates on $U \subset M$, while $i, j = 1, \dots, n$
index the frame elements.

Let $\partial_\mu = \frac{\partial}{\partial x^\mu}$. then we have for $s = \sum a^i s_i$

$$\nabla_{\partial_\mu} s = \sum_i (\partial_\mu a^i) s_i + \sum_{i,j} \underbrace{a^i A_{\mu i}^{\;j}}\, s_j$$

This is the μ-derivative of the vector function $\vec{a} = (a^1, \dots, a^r)$

This is matrix multiplication $A_\mu \cdot \vec{a}$ where $A_\mu = (A_{\mu i}^{\;j})_{i,j=1}^{r}$

Thus we could define an operator D_μ acting on vector valued functions

$$\nabla_\mu \vec{a} = \partial_\mu \vec{a} + A_\mu \cdot \vec{a}$$

Correspondence
$$\vec{a} \longleftrightarrow s = \sum a^i s_i$$
$$\nabla_\mu \vec{a} \longleftrightarrow \nabla_{\partial_\mu} s$$

Since a connection such as $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is tensorial with respect to $X \in \mathfrak{X}(M)$, we can also think of ∇ as a map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$
$$s \longmapsto \nabla s$$

where ∇s is the E-valued 1-form such that $(\nabla s)(X) = \nabla_X s$

Thus $\nabla s = \sum_{\mu=1}^{n} (\nabla_{\partial_\mu} s)\, dx^\mu$ or w.r.t. local frame s_i

$$\nabla \vec{a} = \sum_\mu (\nabla_\mu \vec{a})\, dx^\mu = \sum_\mu \left(\partial_\mu \vec{a}\, dx^\mu + (A_\mu dx^\mu) \cdot \vec{a} \right)$$

$$= d\vec{a} + A \cdot \vec{a}$$

where $d\vec{a} = \sum_\mu (\partial_\mu \vec{a})\, dx^\mu$ is the exterior derivative of a vector-valued function, and $A \cdot \vec{a}$ is the product of the vector-valued function $\vec{a}$ by the matrix-valued 1-form $A = \sum_\mu A_\mu dx^\mu$.

To summarize: With respect to a local frame, a connection always has the form $\nabla = d + A$, where A is a matrix-valued 1-form.

Now we shall express the curvature in similar terms.
The definition is the same as before, but we call it F:

Curvature: $\quad F = F_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$

$$F(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s$$

The same proof as for the Riemann curvature tensor shows that F is tensorial, so it corresponds to a section

$$F \in \Gamma(T^*M \otimes T^*M \otimes E^\vee \otimes E) = \Gamma(T^*M \otimes T^*M \otimes \mathcal{H}om(E,E))$$
$$= \Gamma(T^*M \otimes T^*M \otimes \mathcal{E}nd(E))$$

Since F is obviously skew symmetric in X and Y, we actually have $\quad F \in \Gamma(\Lambda^2 T^*M \otimes \mathcal{E}nd(E)) = \Omega^2(M, \mathcal{E}nd(E))$
that is, the curvature F is a two-form with values in endomorphisms of (the fibers of) E.
With respect to a local frame $\{s_i\}$ an endomorphism of E is just a matrix: $\quad \varphi \in \Gamma(\mathcal{E}nd(E))$, $\quad s = \sum_j a^j s_j$

$$\varphi s_j = \sum_i \varphi^i_j s_i$$

$$\varphi s = \varphi\Big(\sum_j a^j s_j\Big) = \sum_j a^j \varphi s_j = \sum_j \sum_i a^j \varphi^i_j s_i$$

so $\quad \vec{a} = (a^1, \dots, a^r) \overset{\varphi}{\longmapsto} \Big(\sum_j \varphi^i_j a^j\Big)$

Endomorphism $\varphi \Leftrightarrow$ matrix-valued function (φ^i_j)

So w.r.t. local frame $\{s_i\}$, F is a matrix-valued 2-form.
whereas the connection is $\nabla = d + A$, where A is a matrix-valued 1-form. There must be a relationship!

To see what it is, take local coordinates (x^μ), local frame $\{s_i\}$
So that $\nabla \vec{a} = d\vec{a} + A\cdot\vec{a}$

Now F has components $F_{\mu\nu} = F(\partial_\mu, \partial_\nu)$ ($End(E)$-valued)
$F_{\mu\nu} s = F(\partial_\mu,\partial_\nu)s = \nabla_{\partial_\mu}\nabla_{\partial_\nu} s - \nabla_{\partial_\nu}\nabla_{\partial_\mu} s$ (since $[\partial_\mu, \partial_\nu]=0$)

w.r.t. frame: $F_{\mu\nu} s \Leftrightarrow F_{\mu\nu}\vec{a} = \nabla_\mu\nabla_\nu\vec{a} - \nabla_\nu\nabla_\mu\vec{a} = \nabla_\mu\nabla_\nu\vec{a} - (\mu\leftrightarrow\nu)$

$$= \nabla_\mu(\partial_\nu\vec{a} + A_\nu\vec{a}) - (\mu\leftrightarrow\nu)$$

$$= (\partial_\mu + A_\mu)(\partial_\nu\vec{a} + A_\nu\vec{a}) - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu\partial_\nu\vec{a} + \partial_\mu(A_\nu\vec{a}) + A_\mu\partial_\nu\vec{a} + A_\mu A_\nu\vec{a} - (\mu\leftrightarrow\nu)$$

$$= \underbrace{\partial_\mu\partial_\nu\vec{a}}_{\text{symmetric } \mu\leftrightarrow\nu} + (\partial_\mu A_\nu)\vec{a} + \underbrace{A_\nu\partial_\mu\vec{a} + A_\mu\partial_\nu\vec{a}}_{\text{symmetric } \mu\leftrightarrow\nu} + A_\mu A_\nu\vec{a} - (\mu\leftrightarrow\nu)$$

$$= (\partial_\mu A_\nu)\vec{a} + A_\mu A_\nu\vec{a} - (\mu\leftrightarrow\nu)$$

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu)\vec{a}$$

$$\therefore \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$$

<u>Synthesize</u>: the terms $A_\mu A_\nu - A_\nu A_\mu$ form the commutator of the matrices A_μ, A_ν: $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu$

The term $\partial_\mu A_\nu - \partial_\nu A_\mu$ looks like an exterior derivative:

$$A = \sum_\nu A_\nu dx^\nu, \quad dA = \sum_\nu \sum_\mu (\partial_\mu A_\nu)\, dx^\mu \wedge dx^\nu$$

$$= \sum_{\mu<\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu$$

On the other hand $F = \sum_{\mu<\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

So $F = \sum_{\mu<\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu + \sum_{\mu<\nu} [A_\mu, A_\nu]\, dx^\mu \wedge dx^\nu$

$$= dA + [A,A]$$

where $[A,A]$ is the two form: $[A,A](X,Y) = [A(X), A(Y)]$

There is another, more general, bracket operation on matrix-valued 1-forms that yields a matrix-valued 2-form:

given $A, B \in \Gamma(T^*M \otimes \mathrm{End}(E))$ (matrix-valued 1-forms)

form $A \wedge B \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E) \otimes \mathrm{End}(E))$ (2-form with values in (Matrices) $\otimes$ (Matrices))

Then apply the bracket map $[\,,\,]: \mathrm{End}(E) \otimes \mathrm{End}(E) \to \mathrm{End}(E)$ (commutator of endomorphisms / matrices)

the result is $[A \wedge B] \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E))$ (NO COMMA)

(Note that if $[A,B]$ is defined as $[A,B](X,Y) = [A(X), B(Y)]$, it will not always be skew symmetric, ie. not a 2-form).

In components: $A = \sum_\mu A_\mu dx^\mu \quad B = \sum_\nu B_\nu dx^\nu$

$$A \wedge B = \sum_{\mu,\nu} (A_\mu \otimes B_\nu)\, dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} (A_\mu \otimes B_\nu - A_\nu \otimes B_\mu)\, dx^\mu \wedge dx^\nu$$

$$[A \wedge B] = \sum_{\mu<\nu} ([A_\mu, B_\nu] - [A_\nu, B_\mu])\, dx^\mu \wedge dx^\nu$$

In the case $A=B$:

$$[A\wedge A] = \sum_{\mu<\nu} \left([A_\mu, A_\nu] - [A_\nu, A_\mu]\right) dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} \left(2[A_\mu, A_\nu]\right) dx^\mu \wedge dx^\nu$$

$$= 2 \sum_{\mu<\nu} [A_\mu, A_\nu] dx^\mu \wedge dx^\nu = 2[A,A]$$

Thus we have the Cartan Structure equation

$$F = dA + [A,A] = dA + \frac{1}{2}[A\wedge A]$$

Rmk sometimes this is written $F = dA + A\wedge A$. Here $A\wedge A$ is the two form obtained by multiplying the matrix-values together. In our notation this would be $m(A\wedge A)$ where $m: End(E)\otimes End(E) \to End(E)$ is the multiplication map.

Changing the local frame (Gauge transformations)

Since A is a 1-form and F is a 2-form, we know how they transform when we change coordinates $(x^\mu) \to (\tilde{x}^\mu)$ on the base manifold M. How do they change if we change the local trivialization?

Suppose we have two frames $\{s_i\}$, $\{\tilde{s}_i\}$ for E over an open set U with coordinates (x^μ)

we write a general section s as $\begin{cases} s = \sum_i a^i s_i \\ s = \sum_i \tilde{a}^i \tilde{s}_i \end{cases}$

There is some matrix-valued function $G: U \to Mat_{n\times n}$ Such that $\tilde{a} = G^{-1} a$, and $a = G\tilde{a}$

Consider the expressions representing ∇s in the two frames:

w.r.t $\{s_i\}$: $\nabla s \leftrightarrow da + A \cdot a$ $\quad A$ matrix-valued 1-form

w.r.t. $\{\tilde{s}_i\}$: $\nabla s \leftrightarrow d\tilde{a} + \tilde{A}\tilde{a}$ $\quad \tilde{A}$ matrix valued 1-form.

We must have $a = G\tilde{a}$, and $G(d\tilde{a} + \tilde{A}\tilde{a}) = (da + Aa)$
this lets us solve for $\tilde{A}$ in terms of A:

$$G(d\tilde{a} + \tilde{A}\tilde{a}) = d(G\tilde{a}) + AG\tilde{a} = (dG)\tilde{a} + G\,d\tilde{a} + AG\tilde{a}$$

$$Gd\tilde{a} + G\tilde{A}\tilde{a} = (dG)\tilde{a} + G\,d\tilde{a} + AG\tilde{a}$$
$$G\tilde{A}\tilde{a} = (dG)\tilde{a} + AG\tilde{a}$$
$$\tilde{A}\cdot\tilde{a} = (G^{-1}dG)\cdot\tilde{a} + G^{-1}AG\cdot\tilde{a}$$

$$\therefore \quad \tilde{A} = \underbrace{G^{-1}dG}_{\text{fundamental 1-form}} + \underbrace{G^{-1}AG}_{\text{conjugation of } A \text{ by } G}$$

on the Lie group $GL(n,\mathbb{R})$

On the other hand, the curvature F is just an $\mathrm{End}(E)$ valued form, so its matrix representation changes by conjugation when we change the frame:

$$\tilde{F} = G^{-1}FG \qquad \begin{array}{l} F = \text{matrix w.r.t. } \{s_i\} \\ \tilde{F} = \text{matrix w.r.t. } \{\tilde{s}_i\} \end{array}$$

Indeed this is compatible with the structure equation. (HW)