Lecture 18

Sectional, Ricci, Scalar curvature; Einstein Field Equations.

Recall $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

in coordinates $\quad R = \sum R_{ijk}{}^{\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^\ell}$

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \sum_\ell R_{ijk}{}^{\ell} \frac{\partial}{\partial x^\ell}$$

<u>Sectional curvature</u>: If $v, w \in T_pM$ are orthonormal vectors, the sectional curvature in the plane spanned by v and w is

$$K_p(v,w) := g\big(R(v,w)w, v\big) \qquad \big(g(v,w)=0,\ \|v\|=\|w\|=1\big)$$

If v and w are not assumed orthonormal, the formula is

$$K_p(v,w) := \frac{g(R(v,w)w,v)}{g(v,v)g(w,w) - g(v,w)^2}$$

The symmetries of the curvature tensor imply that the sectional curvature does not depend on the choice of basis, only on the point p and the plane in the tangent space spanned by v, w.

<u>Traces of tensors</u>: If $T: V \to V$ is a linear transformation, it has a well-defined trace: pick a basis $\{e_1, \dots, e_n\}$ of V, let T_i^j be the components so that $T(e_i) = \sum_j T_i^j e_j$, and set

$$tr(T) := \sum_{i=1}^n T_i^i \qquad (\text{sum of "diagonal" components})$$

This does not depend on the choice of basis.

If $g: V\times V \to \mathbb{R}$ is a nondegenerate inner product, and $B: V\times V\to\mathbb{R}$ is any bilinear form, then B has a trace with respect to g.

$$B(x,y) = g(x, \tilde{B}(y)) \qquad \text{for a unique map } \tilde{B}: V\to V$$

$$\text{define: } \quad tr_g(B) := tr\,\tilde{B}.$$

Pick basis $e_1, e_2, \dots, e_n$: then $B_{ij} = B(e_i, e_j) \quad g_{ij} = g(e_i, e_j)$

$$\text{and} \quad B_{ij} = g(e_i, \tilde{B}(e_j)) = \sum_k g_{ik} \tilde{B}^k_j$$

$$\text{So} \quad \tilde{B}^k_j = \sum_i g^{ik} B_{ij} \qquad (g^{ik} = \text{inverse matrix of } g_{ik})$$

$$tr_g\, B = \sum_j \tilde{B}^j_j = \sum_{i,j} g^{ij} B_{ij}$$

Or, if we take $e_1, \dots, e_n$ to be a g-orthonormal basis of V, then

$$tr_g\, B = \sum_i B_{ii} = \sum_i B(e_i, e_i)$$

<u>Ricci curvature.</u> Define a bilinear form $Ric: T_pM \times T_pM \to \mathbb{R}$

$$Ric(X,Y) = tr\,(V \mapsto R(V,X)Y)$$

Ric is called Ricci curvature. The symmetries of R imply that it is a symmetric bilinear form on each tangent space.

Interpretation: $\frac{1}{n-1} Ric(x,x)$ is the "average" sectional curvature of all two planes containing x.

In coordinates $\quad Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = Ric_{ij} = \sum_{k=1}^{n} R_{kij}{}^{k}$

[I've noticed that this formula does not match standard notation, where $\quad Ric_{ab} = \sum_c R^c{}_{acb}$. It's due to a different notation for Riemann curvature.]

Scalar curvature: Define a function $s: M \to \mathbb{R}$ by

$$s = tr_g(Ric)$$

so $s(p) = \sum_i Ric_p(e_i, e_i)$ where e_i is an orthonormal basis of T_pM.

or $s = \sum_{i,j} g^{ij} Ric_{ij} = \sum_{i,j,k} g^{ij} R_{kij}{}^{k}$

Roughly, $s(p)$ is an "average" of all sectional curvatures at p.

Importance: • The sectional curvature is essentially equivalent to the full Riemann curvature tensor, in the sense that if we know all sectional curvatures of all two planes at all points, the Riemann curvature tensor can be determined.

• Ricci and scalar curvature carry progressively less information, but they have geometric meaning: they are related to the deviation of volume from the Euclidean case. (cf. Bishop's theorem)

• A very important combination of tensors is the Einstein tensor

$$G = Ric - \frac{1}{2} s g \qquad G_{ij} = Ric_{ij} - \frac{1}{2} s g_{ij}$$

The fundamental equation of General Relativity is

$$\underbrace{Ric - \frac{1}{2} s g}_{\text{Einstein tensor}} + \Lambda g = 8\pi T$$

stress-energy tensor

Λ = cosmological constant

Here the metric has signature $(-+++)$

- If Λ and T are taken to be zero, and $n \neq 2$, then the equation reduces to

$$Ric = 0.$$

So the metric is Ricci-flat.

- If $T=0$ but Λ is not, the equations reduce to

$$Ric = \lambda g \quad (\lambda \text{ constant})$$

Such metrics are called Einstein metrics, and the are much studied in the positive definite case as well.

Reason: Take trace of $Ric - \frac{1}{2} s g + \Lambda g = 0$

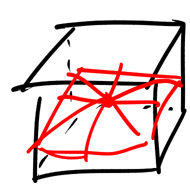
$tr_g\, g = n \qquad tr_g\, Ric = s \qquad$ so

$$s - \frac{1}{2} s n + \Lambda n = 0 \iff s\left(1 - \frac{n}{2}\right) + \Lambda n = 0$$

$\Rightarrow s$ is constant and $s = \dfrac{n\Lambda}{\left(\frac{n}{2} - 1\right)}$.

Interpretation of sectional curvature: For $v, w \in T_pM$, let $V = \mathrm{span}(v,w) \subset T_pM$ then $K(v,w)$ is the Gaussian curvature at p of the surface

$$\exp(V) \subset M$$



exp(V).

Model Spaces of constant sectional curvature

$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, g = induced metric, $K > 0$

$\mathbb{R}^n$, g = standard $K = 0$

$\mathbb{H}^n = \{(x_1, \dots, x_n) \mid x > 0\}$ $\quad g = \dfrac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$

Bishop's Volume Theorem: Suppose (M,g) satisfies $\mathrm{Ric} \geq \rho$ (ρ constant) (meaning that $\mathrm{Ric}(x,x) \geq \rho$ for every unit vector x.) Let $E(\rho)$ be the model space of constant curvature whose Ricci curvature equals ρ. (Either $S^n, \mathbb{R}^n, \mathbb{H}^n$ with rescaled metric) Then

$$\mathrm{Vol}(B_M(x,r)) \leq \mathrm{Vol}(B_{E(\rho)}(r))$$

geodesic ball of radius r in M

geodesic ball in model space

Now, if $\rho > 0$, then the model space is S^n, which is compact and has finite volume. Thus if $\mathrm{Ric} \geq \rho > 0$ and (M,g) is complete, M has finite volume. In fact:

Thm (Bonnet-Myers) if $\mathrm{Ric} \geq \frac{1}{r^2} > 0$ and (M,g) is complete then M is compact and $\mathrm{diam}(M) \leq \pi r$

Cor M has finite fundamental group

Proof The universal cover $\tilde{M}$ also satisfies $\mathrm{Ric} \geq \frac{1}{r^2} > 0$, and so $\tilde{M}$ is compact. Hence the covering map $\tilde{M} \to M$ can only have finitely many sheets.