Riemann curvature as an obstruction

(M, g, ∇) as usual

- Recall $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$
- We say (M,g) is locally isometric to Euclidean space if $\forall p \in M$, $\exists$ U open, $p \in U \subset M$ and $V \subset \mathbb{R}^n$ open and an isometry $f: U \to V$. (V is given standard Euclidean metric).
- We knew that if (M,g) is locally isometric to Euclidean space, then $R \equiv 0$. The converse is also true.

<u>Theorem</u>: Suppose that (M,g) satisfies $R \equiv 0$. (Such a manifold is called flat.) Then (M,g) is locally isometric to Euclidean space.

The zero curvature conditions is one of the prime examples of an integrability condition.

<u>Some preliminaries</u>: Since the result is local, we can work in local coordinates $(x^1, \dots, x^n)$. We can rewrite the definition of curvature (using $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$, with $e_i = \frac{\partial}{\partial x^i}$)

$$\nabla_{e_i} \nabla_{e_j} Z - \nabla_{e_j} \nabla_{e_i} Z = R(e_i, e_j) Z$$

or $$[\nabla_{e_i}, \nabla_{e_j}] = R(e_i, e_j)$$

where ∇_{e_i}, ∇_{e_j}, $R(e_i, e_j)$ are operators on vector fields.

So $R(e_i, e_j)$ measures the failure of covariant derivatives in different directions to commute.

If $\frac{D}{\partial x^i} Z = \nabla_{e_i} Z$, then $\frac{D}{\partial x^i} \frac{D}{\partial x^j} Z - \frac{D}{\partial x^j} \frac{D}{\partial x^i} Z = R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) Z$

Corollary: If $R \equiv 0$, then $\nabla_{e_i} \nabla_{e_j} Z = \nabla_{e_j} \nabla_{e_i} Z$

$$\frac{D}{\partial x^i} \frac{D}{\partial x^j} Z = \frac{D}{\partial x^j} \frac{D}{\partial x^i} Z$$

Lemma 1: If $R \equiv 0$, and $v \in T_pM$ is any vector, then locally near p there is a vector field Z such that $Z(p) = v$ and $\nabla_X Z = 0$ for all vectors X.

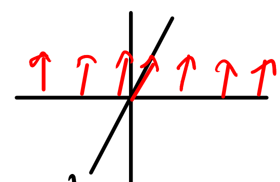
Such a vector field Z is called covariantly constant.

Proof: Work in local coordinates $(x^1, \dots, x^n)$ so that $p = (0, 0, \dots, 0)$

Define $Z(0, \dots, 0) = v$. First we define Z along the x^1-axis, then along the x^1x^2-plane, then along the $x^1x^2x^3$ subspace, and so on.

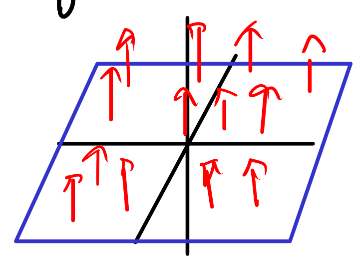
Define $Z(x^1, 0, \dots, 0)$ as the parallel transport of v along the x^1-axis. Thus by definition

$$\frac{D}{\partial x^1} Z(x^1, 0, \dots, 0) = \nabla_{e_1} Z(x^1, 0, \dots, 0) = 0$$



Now to define $Z(x^1, x^2, 0, \dots, 0)$, use parallel transport along lines parallel to the x^2-axis. Then by definition

$$\frac{D}{\partial x^2} Z(x^1, x^2, 0, \dots, 0) = \nabla_{e_2} Z(x^1, x^2, 0, \dots, 0) = 0$$



Now we claim that $\frac{D}{\partial x^1} Z(x^1, x^2, 0, \dots, 0) = 0$

By construction $\frac{D}{\partial x^1} Z(x^1, x^2, 0, \dots, 0) = 0$ holds when $x^2 = 0$

Now we have $\frac{D}{\partial x^2} \frac{D}{\partial x^1} Z(x^1, x^2, 0, \dots, 0) = \frac{D}{\partial x^1} \frac{D}{\partial x^2} Z(x^1, x^2, 0, \dots, 0) = \frac{D}{\partial x^1}(0) = 0$

Thus $\frac{D}{\partial x^1} Z$ vanishes when $x^2 = 0$, and its derivative w.r.t. x^2 vanishes, so it vanishes for all values of x^2

To define $Z(x^1, x^2, x^3, 0, \dots)$ use parallel transport along lines parallel to x^3-axis. Then

$$\frac{D}{\partial x^3} Z(x^1, x^2, x^3, 0, \dots) = 0 \quad \text{by construction}$$

Now $\frac{D}{\partial x^3}\left(\frac{D}{\partial x^1} Z\right) = \frac{D}{\partial x^1}\frac{D}{\partial x^3} Z = 0$ and $\frac{D}{\partial x^1} Z = 0$ when $x^3 = 0$

$\frac{D}{\partial x^3}\left(\frac{D}{\partial x^2} Z\right) = \frac{D}{\partial x^2}\frac{D}{\partial x^3} Z = 0$ and $\frac{D}{\partial x^2} Z = 0$ when $x^3 = 0$.

Thus we conclude $\frac{D}{\partial x^1} Z = 0$, $\frac{D}{\partial x^2} Z = 0$, and $\frac{D}{\partial x^3} Z = 0$

for all points $(x^1, x^2, x^3, 0, \dots, 0)$

Proceeding in this way, we construct $Z(x^1, \dots, x^k, 0, \dots, 0)$ such that $\frac{D}{\partial x^j} Z(x^1, \dots, x^k, 0, \dots, 0) = 0$ for $j = 1, \dots, k$

Finally we get $Z(x^1, \dots, x^n)$ such that

$\frac{D}{\partial x^j} Z = \nabla_{e_j} Z = 0$ for $j = 1, \dots, n$

Then for any vector field $X = \sum_{j=1}^{n} X^j e_j$, $\nabla_X Z = \sum_{j=1}^{n} X^j \nabla_{e_j} Z = 0$. ▨

Lemma 2: If Z_1 and Z_2 are vector fields such that $\nabla_X Z_1 = \nabla_X Z_2 = 0$ for all X, then $[Z_1, Z_2] = 0$.

Proof this use the torsion-free property of ∇.

$$\underbrace{\nabla_{Z_1} Z_2 - \nabla_{Z_2} Z_1}_{\text{this vanishes by hypothesis.}} = [Z_1, Z_2]$$

▨

Proof of theorem: Let $v_1, \dots, v_n$ be an orthonormal basis of T_pM.

$$g(v_i, v_j) = \delta_{ij}$$

Apply lemma 1: This yields a collection of vectorfields $Z_1, \dots, Z_n$ defined in a neighborhood of p such that $Z_i(p) = v_i$ and $\nabla_X Z_i = 0$ for all X.

Since parallel transport preserves the metric [Homework] we find $g(Z_i, Z_j) = g(v_i, v_j) = \delta_{ij}$ at all points where Z_i, Z_j are defined. That is $Z_1, \dots, Z_n$ are an orthonormal basis of each tangent space.

Now apply Lemma 2: We get $[Z_i, Z_j] = 0$.
We claim there is a coordinate system $(y^1, \dots, y^n)$ near p such that $\frac{\partial}{\partial y^i} = Z_i$.

First take the basis of 1-forms α_i $(i=1,\dots,n)$ dual to Z_i so $\alpha_i(Z_j) = \delta_{ij}$. The coordinate free formula for exterior derivative

$$d\alpha(X,Y) = X.\alpha(Y) - Y.\alpha(X) - \alpha([X,Y])$$

implies that each α_i is closed:

$$d\alpha_i(Z_j, Z_k) = Z_j.\alpha_i(Z_k) - Z_k\alpha_i(Z_j) - \alpha_i([Z_j, Z_k])$$
$$= Z_j.\delta_{ik} - Z_k.\delta_{ij} - \alpha(0) = 0$$

Since α_i is closed, it is locally exact. There is y^i such that $dy^i = \alpha_i$. [We can choose the constant of integration such that $y^i(p) = 0$.] Since the Z_i's are a basis of the tangent space at each point, the α_i's are a basis of the cotangent space at each point, and so $(y^1, \dots, y^n)$ is a collection of functions with linearly independent differentials.

Hence $(y^1, \dots, y^n)$ is a coordinate system.
Also $\frac{\partial}{\partial y^i}$ is dual to $dy^i = \alpha_i$, so by construction $\frac{\partial}{\partial y^i} = Z_i$.

At last, in the coordinate system $(y^1, \dots, y^n)$, the metric has components $g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g(Z_i, Z_j) = \delta_{ij}$

Thus the metric is Euclidean in these coordinates. ☐

We used the fact that for a 1-form α

$$d\alpha(X,Y) = X.\alpha(Y) - Y.\alpha(X) - \alpha([X,Y])$$

<u>Proof</u> since both sides are $\mathbb{R}$-linear in α and local, it suffices to check for a 1-form $\alpha = f\,dg$ $\quad f, g$ local functions

$\alpha = f\,dg \qquad d\alpha = df \wedge dg$

$$LHS = d\alpha(X,Y) = (df \wedge dg)(X,Y) = df(X)dg(Y) - dg(X)df(Y)$$
$$= (X.f)(Y.g) - (X.g)(Y.f)$$

$$RHS = X.\alpha(Y) - Y.\alpha(X) - \alpha([X,Y])$$
$$= X.(f(Y.g)) - Y.(f(X.g)) - f([X,Y].g)$$

$$= (X.f)(Y.g) + \underline{f\,X.(Y.g)} - (Y.f)(X.g) - \underline{f\,Y.(X.g)} - \underline{f([X,Y].g)}$$

these cancel

<u>Symmetries of R</u>: $\quad R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

(1) $\quad R(X,Y)Z = -R(Y,X)Z \quad$ i.e. R is skew symmetric in X and Y.
this is obvious.

(2) (First Bianchi identity, not due to Bianchi)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Consider the expression $g(R(X,Y)Z, W)$ which is a tensorial map
$g(R(-,-)-,-) : \mathfrak{X}(M)^4 \to C^\infty(M)$
this expression has components

$$g(R(e_i,e_j)e_k, e_l) = g\left(\sum_m R_{ijk}{}^m e_m, e_l\right) = \sum_m g_{ml} R_{ijk}{}^m$$

We define $R(X,Y,Z,W) = g(R(X,Y)Z, W)$

components : $R_{ijkl} = \sum_m g_{ml} R_{ijk}{}^m$

This is called the "fully covariant" Riemann curvature tensor.
It is an example of "lowering an index".

We have

(3) $$g(R(X,Y)Z,W) = g(R(X,Y)W,Z)$$

(skew symmetry in $Z \leftrightarrow W$)

As a consequence of (1), (2), (3), we have (4)

(4) $$g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$$

(symmetry in $(X,Y) \leftrightarrow (Z,W)$)

It is possible to show that the space of tensors (at a point) having the symmetries (1)-(4) has dimension

$$\frac{n^2(n^2-1)}{12}.$$