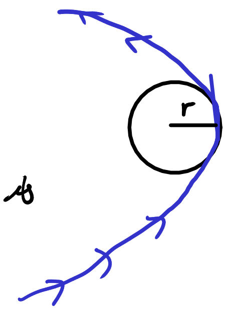
Curvature

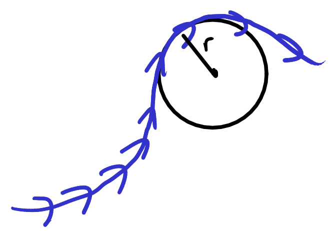
Today we will introduce the Riemann curvature tensor. But first some history of this important concept.

Curvature of a curve in a plane is defined to be the reciprocal of the radius of the osculating circle. We can associate a $\pm$ sign depending on an orientation of the curve.

$k = \frac{1}{r}$



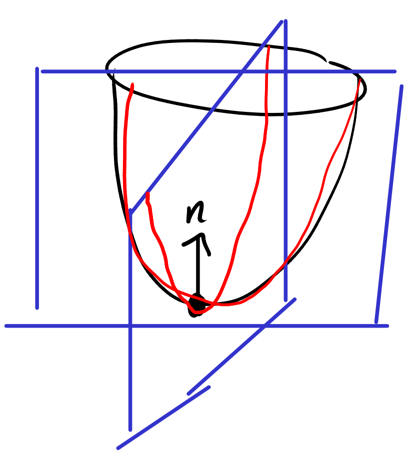
$k > 0$ since circle is on left side



$k = -\frac{1}{r}$

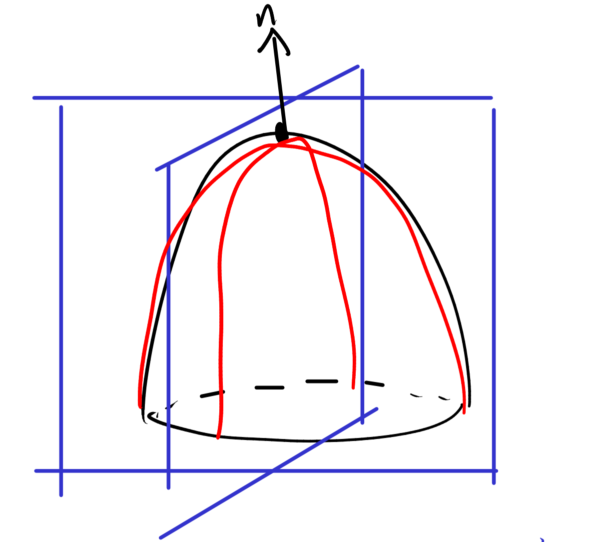
$k < 0$ since circle is on right side

Gaussian Curvature of a surface in 3-space let $\vec{n}$ denote the normal vector to the surface S. Consider cutting S by a plane containing $\vec{n}$ (a normal plane). Then we get a curve in a plane an we can consider its curvature (Sign now depends on whether the osculating circle sits above or below the surface) Let $K = k_{max} \cdot k_{min}$, where k_{min} and k_{max} are the maximum and minimum values obtained over all choices of normal plane.



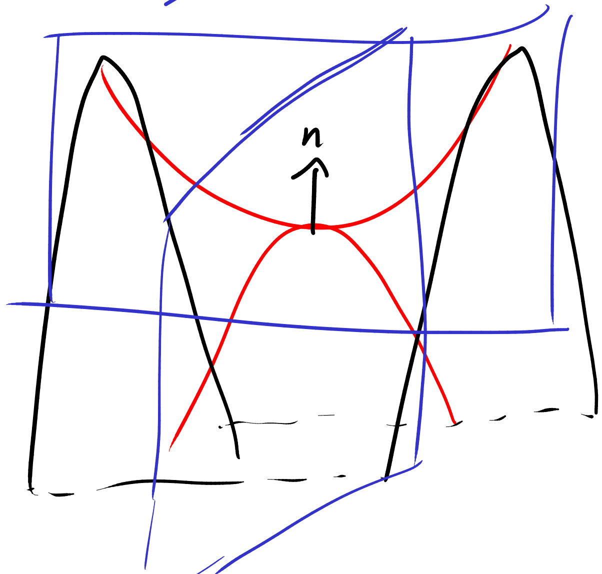
$k_{min} > 0 \qquad k_{max} > 0 \qquad K > 0$

convexo-convex



$k_{min} < 0 \quad k_{max} < 0 \quad K < 0$

concavo-concave

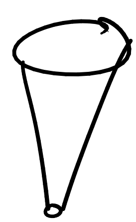


$k_{min} < 0 \quad k_{max} > 0 \quad K < 0$

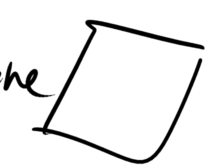
concavo-convex

Gauss showed that this measure of curvature depends only on the metric induced on the surface, not on the embedding of the surface in $\mathbb{R}^3$. (Theorema Egregium)

For example
the cone minus the vertex



and the flat plane



are locally isometric, so they must have the same curvature, zero in this case.

How does Gauss prove this? If (p,q) are local coordinates the metric looks like $g = E\,dp\otimes dp + F(dp\otimes dq + dq\otimes dp) + G\,dq\otimes dq$

Gauss finds a formula for K in terms of E, F, G, their first derivative, and their second derivatives. This formula is <u>extremely</u> <u>complicated</u>.

Riemann saw that this formula could be generalized to higher dimensions. He introduced an intrinsic notion of curvature as the obstruction to the existence of a local isometry with Euclidean space.

We now proceed to give a modern treatment.
Let (M,g) be a Riemannian manifold; ∇ is its Levi-Civita connection
$\mathfrak{X}(M)$ = vector fields on M.

Def The Riemann curvature is the function

$$\begin{aligned} R : \mathfrak{X}(M)^3 &\longrightarrow \mathfrak{X}(M) \\ (X,Y,Z) &\longmapsto R(X,Y)Z \end{aligned}$$

$$R(X,Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

Prop The Riemann curvature is tensorial, so it corresponds to a section
$$R \in \Gamma(M, (T^*M)^{\otimes 3} \otimes TM).$$

Proof We must check that R is $C^\infty(M)$-linear in each factor
This isn't obvious because X, Y, Z all get differentiated somehow.
But the formula is set up so that things cancel.

Claim 1: $R(fX,Y)Z = f\,R(X,Y)Z$

$$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z$$

$$= f\nabla_X \nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{[fX,Y]} Z$$

$$= f\nabla_X\nabla_Y Z - (Y.f)\nabla_X Z - f\nabla_Y\nabla_X Z - \nabla_{[fX,Y]} Z$$

Now $[fX,Y] = f[X,Y] - (Y.f)X$ (Check in local coords)

So $\nabla_{[fX,Y]} Z = \nabla_{(f[X,Y]-(Y.f)X)} Z = \nabla_{f[X,Y]} Z - (Y.f)\nabla_X Z$

$$= f\nabla_{[X,Y]} Z - (Y.f)\nabla_X Z$$

$\therefore$ $R(fX,Y)Z$

$$= f\nabla_X\nabla_Y Z - (Y.f)\cancel{\nabla_X Z} - f\nabla_Y\nabla_X Z - \left(f\nabla_{[X,Y]} Z - (Y.f)\cancel{\nabla_X Z}\right)$$

$$= f\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z\right) = fR(X,Y)Z$$

which was the claim 1

Claim 2: $R(X,fY)Z = fR(X,Y)Z$ follows from claim 1 by the obvious skew symmetry $R(Y,X)Z = -R(X,Y)Z$

Claim 3 $R(X,Y)(fZ) = fR(X,Y)Z$

$$\nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ)$$

$$= \nabla_X\left((Y.f)Z + f\nabla_Y Z\right) - \nabla_Y\left((X.f)Z + f\nabla_X Z\right) - \left(([X,Y].f)Z + f\nabla_{[X,Y]} Z\right)$$

$$= X.(Y.f)Z + (Y.f)\nabla_X Z + (X.f)\nabla_Y Z + f\nabla_X\nabla_Y Z$$

$$-\left\{Y.(X.f)Z + (X.f)\nabla_Y Z + (Y.f)\nabla_X Z + f\nabla_Y\nabla_X Z\right\}$$

$$-\left\{([X,Y].f)Z + f\nabla_{[X,Y]} Z\right\}$$

cancel in pairs

cancel since $[X,Y].f = X(Y.f) - Y(X.f)$

Thus R is called the Riemann curvature tensor.
Note that R is determined by the Levi-Civita connection, which is determined by the metric g. Hence if two manifolds are isometric then their curvatures are equal. And finally note that the curvature of Euclidean space is zero. So if a manifold has nonzero R it cannot be isometric to Euclidean space, even locally.

Coordinate representation of R: Let $(x^1, \ldots, x^n)$ be local coordinates

$\frac{\partial}{\partial x^i}$ coordinate vector fields

dx^i coordinate 1-forms

Since R is a tensor and a section of $(T^*M)^{\otimes 3} \otimes TM$, it can be written in components

$$R = \sum_{i,j,k,l=1}^{n} R_{ijk}^{l} \, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$$

where R_{ijk}^{l} is the $\frac{\partial}{\partial x^l}$-component of $R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}$

Abbreviate $e_i = \frac{\partial}{\partial x^i}$ then $g_{ij} = g(e_i, e_j)$

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} e_k \quad \text{Christoffel symbolse}$$

We compute $R(e_i, e_j)e_k = \nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k$

But $[e_i, e_j] = 0$ since coordinate vector fields commute.

So $R(e_i,e_j)e_k = \nabla_{e_i}(\nabla_{e_j}e_k) - \nabla_{e_j}(\nabla_{e_i}e_k)$

$$= \nabla_{e_i}\Big(\sum_\ell \Gamma^\ell_{jk} e_\ell\Big) - \nabla_{e_j}\Big(\sum_\ell \Gamma^\ell_{ik} e_\ell\Big)$$

$$= \sum_\ell \Big(\frac{\partial}{\partial x^i}(\Gamma^\ell_{jk})e_\ell + \Gamma^\ell_{jk}\nabla_{e_i}e_\ell\Big) - (\text{same with } i,j \text{ swapped})$$

$$= \sum_\ell \Big(\frac{\partial \Gamma^\ell_{jk}}{\partial x^i} e_\ell + \Gamma^\ell_{jk}\sum_m \Gamma^m_{i\ell} e_m\Big) - (i \leftrightarrow j)$$

$$= \sum_m \Big(\frac{\partial \Gamma^m_{jk}}{\partial x^i} + \sum_\ell \Gamma^\ell_{jk}\Gamma^m_{i\ell}\Big)e_m - (i \leftrightarrow j)$$

$$= \sum_m \Big(\frac{\partial \Gamma^m_{jk}}{\partial x^i} - \frac{\partial \Gamma^m_{ik}}{\partial x^j} + \sum_\ell \big(\Gamma^\ell_{jk}\Gamma^m_{i\ell} - \Gamma^\ell_{ik}\Gamma^m_{j\ell}\big)\Big)e_m$$

Since R^m_{ijk} is the coefficient of e_m, we find

$$R^m_{ijk} = \frac{\partial \Gamma^m_{jk}}{\partial x^i} - \frac{\partial \Gamma^m_{ik}}{\partial x^j} + \sum_\ell \big(\Gamma^\ell_{jk}\Gamma^m_{i\ell} - \Gamma^\ell_{ik}\Gamma^m_{j\ell}\big)$$

So the curvature involves the Christoffel symbols and their first derivatives. But now recall that

$$\Gamma^k_{ij} = \sum_\ell \frac{1}{2} g^{k\ell}\Big(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\Big)$$

involves the metric g and its first derivatives. $[(g^{k\ell}) = (g_{ij})^{-1}]$
So R involves 2nd derivatives of the metric, just like Gauss' formula for curvature of surfaces!