Hopf-Rinow theorem

Now we will address whether any to points in a Riemannian manifold can be joined by a shortest path.
Recall the exponential map $\exp : TM \to M$, and the distance function $\mathrm{dist}_g : M\times M \longrightarrow \mathbb{R}$, $\mathrm{dist}_g(p,q) = \inf \left\{ \ell(\gamma) \;\middle|\; \begin{array}{l} \gamma(0)=p,\ \gamma(1)=q \\ \gamma \text{ a piecewise smooth path} \end{array} \right\}$

<u>Def</u> (M,g) is geodesically complete if every geodesic can be extended for all time. In other words, the exponential map $\exp_q(v)$ is defined for all $(q,v)\in TM$.

<u>Recall Def</u> (M,d) is complete as a metric space if every Cauchy sequence has a limit.

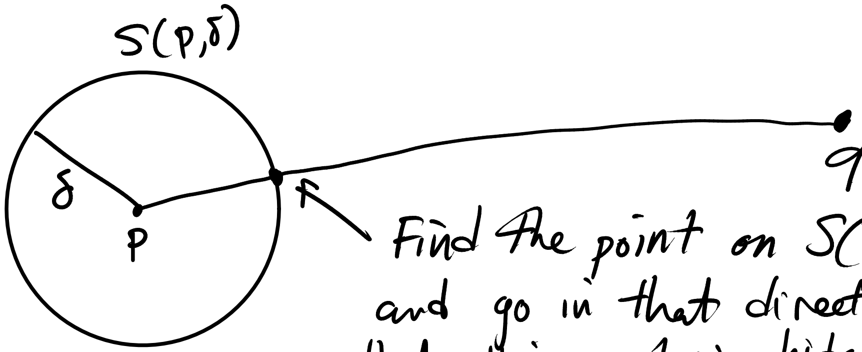
<u>Theorem (Hopf-Rinow)</u> (M,g) is geodesically complete if and only if (M, dist_g) is complete as a metric space.
Moreover, if (M,g) is geodesically complete then any two points p,q may be joinded by a geodesic whose length equals $\mathrm{dist}_g(p,q)$. (That is, minimizers always exist.)

<u>Proof</u> Part 1: Geodesically complete $\Longrightarrow$ minimizers exist.
Let $p,q\in M$, and set $r = \mathrm{dist}_g(p,q)$.
Let U_p be an open set around p and $\varepsilon>0$ such that

$$\exp_p : \{ v\in T_pM \mid \|v\|<\varepsilon\} \longrightarrow U_p \quad \text{is a diffeo.}$$

(This exists by results of previous lecture).

Pick $\delta<\varepsilon$, and consider $S=S(p,\delta)$ the spherical shell



Find the point on $S(p,\delta)$ closest to q, and go in that direction. We need to see that this geodesic hits q.

More precisely: The function $dist_g(q,-) : S \longrightarrow \mathbb{R}$ achieves its minimum at some $p_0 \in S$ (since it is a continuous function on a compact set).

Thus $p_0 = \exp_p(\delta v)$ for $v \in T_pM$ $\|v\| = 1$
and $dist_g(p_0, q) \leq dist_g(s,q)$ $\forall s \in S$.

<u>Claim (*)</u>: $\exp_p(rv) = q$ (This will complete part 1)

Set $\gamma(t) = \exp_p(tv)$. It exists for all t by geodesic completeness
The previous claim is a special case of

<u>Claim (**)</u>: $dist(\gamma(t), q) = r - t$ for $t \in [\delta, r]$
(take $t = r$ to get (*))

Proof of (**) for $t = \delta$: Need to show $dist(p_0, q) = r - \delta$
Well

$$r = dist(p,q) = \inf_{s \in S} \left(dist(p,s) + dist(s,q) \right)$$ since every path touches S.

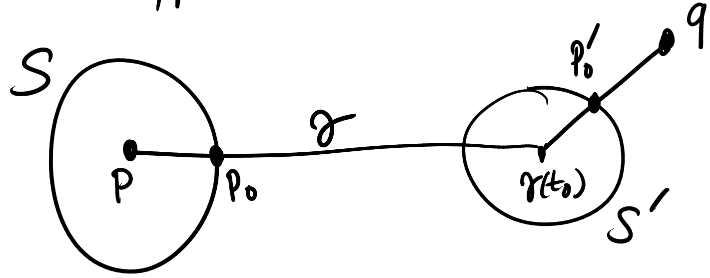
$$= \inf_{s \in S} \left(\delta + dist(s,q) \right) = \delta + \inf_{s \in S} dist(s,q)$$

$$= \delta + dist(p_0, q)$$ since p_0 is closest point on S to q.

Thus $dist(p_0, q) = r - \delta$ ✓

Now consider the set I of $t \in [\delta, r]$ for which $(**)$ holds. We know $\delta \in I$, so $I \neq \emptyset$. It is also clear that I is closed, because $(**)$ is equivalent to the vanishing of the continuous function $\operatorname{dist}(\gamma(t), q) - (r-t)$.
If we can show that I is also open, we will know that $I = [\delta, r]$, since the only nonempty open and closed subset of a connected space is the whole space itself.
[This sort of argument is called the "continuity method"].

For openness, suppose $(**)$ holds for $t = t_0$



Let S' be shell of radius δ' around $\gamma(t_0)$. Let $p_0' \in S'$ be point on S' closest to q.

Then $\operatorname{dist}(\gamma(t_0), q) = \delta' + \operatorname{dist}(p_0', q)$ like before

But $\operatorname{dist}(\gamma(t_0), q) = r - t_0$ by assumption.
So
$$\operatorname{dist}(p_0', q) = r - t_0 - \delta' = r - (t_0 + \delta')$$

By triangle inequality
$$\operatorname{dist}(p, p_0') \geq \operatorname{dist}(p, q) - \operatorname{dist}(p_0', q) = r - (r - (t_0 + \delta')) = t_0 + \delta'$$

But the path $p \to q$ obtained from following γ $p \to \gamma(t_0)$ and then the radial geodesic $\gamma(t_0) \to p_0'$ has length $t_0 + \delta'$

It is therefore an absolutely minimal path between its endpoints and so it is a geodesic. Since part of this geodesic coincides with γ, the whole thing coincides with γ.

We find $\gamma(t_0+\delta') = p_0'$,
so $\mathrm{dist}(\gamma(t_0+\delta'), q) = r - (t_0+\delta')$, so $t_0+\delta' \in I$.

In summary, we have shown the following facts about $I \subset [\delta, r]$:

- $\delta \in I$, so $I \neq \emptyset$
- I is closed
- If $t_0 \in I$, then $t_0+\delta' \in I$ for small $\delta' > 0$.

These properties imply $I = [\delta, r]$, so proof of (**) is complete.
This completes part 1.

<u>Part 2</u> Geodesically complete $\Rightarrow$ complete as a metric space.
Suppose $A \subset M$ is a bounded subset of diameter D with respect to dist_g. Then because any two points are joined by a minimizing geodesic we have

$$\text{for } p \in A, \quad A \subset \exp_p\left(\{v \in T_pM \mid \|v\| \leq D\}\right)$$

Since $\{v \in T_pM \mid \|v\| \leq D\}$ is compact and $\exp_p$ is continuous, A is contained in a compact set. Hence $\overline{A}$ is contained in the same compact set, and is therefore itself compact. So (M, dist_g) has the property that any bounded set has compact closure (which is stronger than completeness).

Part 3 Completeness $\Rightarrow$ Geodesic completeness

Let $(p,v) \in TM$, $\|v\|=1$. consider the path $\gamma(t) = \exp_p(tv)$
This is defined on some interval $I \subset [0,\infty)$.
By fundamental existence theorem for geodesics, I is nonempty and open.
We want to show it is also closed.
Let $b \in \overline{I}$ take $t_n \in I$ such that $t_n \to b$
Since $\mathrm{dist}(\gamma(t_n), \gamma(t_m)) = |t_m - t_n|$ for n,m large,

The sequence $\gamma(t_n)$ is Cauchy, and converges to $q \in M$.
It is possible to show $q = \exp_p(bv)$, so $b \in I$. $\blacksquare$

apply theorem on minimizing property of geodesics in this ball.