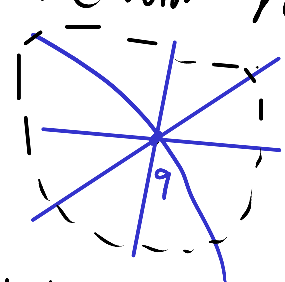
The length-minimizing property of geodesics.

Recall exponential map $\exp : \mathcal{U} \to M$, domain $\mathcal{U} \subset TM$

$$(q,v) \mapsto \exp_q(v)$$

- $\exp_q(v) = \gamma(1)$ where $\gamma(t)$ is a geodesic with $\gamma(0)=q$ $\dot\gamma(0)=v$.
- The geodesics starting at $q \in M$ are all of the form $\gamma(t) = \exp_q(tv)$ for various tangent vectors $v \in T_qM$.



<u>Prop</u>: the length of $\gamma(t) = \exp_q(tv)$ between $t=a$ and $t=b$ is

$$\ell\left(\exp_q(tv)\big|[a,b]\right) = (b-a)\|v\|_g$$

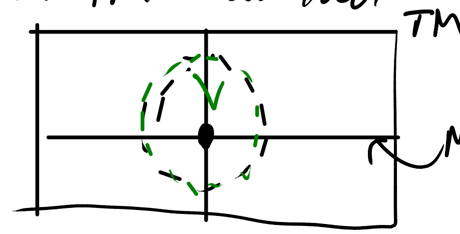
<u>Proof</u> The <u>speed</u> of γ is $\|\dot\gamma\|_g$, and this is constant.

$$\frac{d}{dt}\left(\|\dot\gamma\|_g^2\right) = \frac{d}{dt} g(\dot\gamma,\dot\gamma) = 2g\left(\frac{D}{dt}(\dot\gamma), \dot\gamma\right) = 0$$

Since $\dot\gamma(0) = v$, $\|\dot\gamma(t)\|_g = \|v\|_g$ for all t.

So $\ell = \int_a^b \|\dot\gamma\|_g \, dt = \int_a^b \|v\|_g \, dt = (b-a)\|v\|_g$ ▨

Now recall the ODE theorem, which says that for any compact subset $K \subset TM$, there is an $\varepsilon > 0$ such that $\exp_q(\varepsilon v)$ exists for any $(q,v) \in K$. This implies that for any $p \in M$, there is a neighborhood V of $(p,0)$ in TM such that the exponential map is defined on V.



(Proof: let R = compact subset of M containing p; take

$$K = \{(q,v)\in TM \mid q\in R,\ \|v\|_g \le 1\},$$ which is compact;

Find $\varepsilon>0$ as above; set $V = \{(q,v)\in TM \mid q\in \mathrm{Int}(R),\ \|v\|_g < \varepsilon\}$)

Differential topology of exp: The main thing to see next is that sufficiently close points in M can be joined by a unique shortest geodesic. (that is, a path which is shortest among geodesics. Later we show that this is in fact the shortest path.)

Lemma For each $q\in M$, there is an open set $U_q \subset T_qM$ such that $0\in U_q$ and $\exp_q : U_q \to M$ is a diffeomorphism onto its image.

Proof: By the inverse function theorem, it suffices to check that the derivative $D(\exp_q)_0 : T_0(T_qM) \to T_qM$ is an isomorphism.

An element of $T_0(T_qM)$ is just a tangent vector $v\in T_qM$. Since $\exp_q$ maps the path $(t\to tv)$ to $(t\mapsto \exp_q(tv))$ we see

$$D(\exp_q)_0(v) = \frac{d}{dt}(\exp_q(tv))\Big|_{t=0} = v$$

by construction of exp.

In conclusion, $D(\exp_q)_0$ is the identity map, suitably interpreted. In particular it is an isomorphism. ∎

Taking this argument further, we get

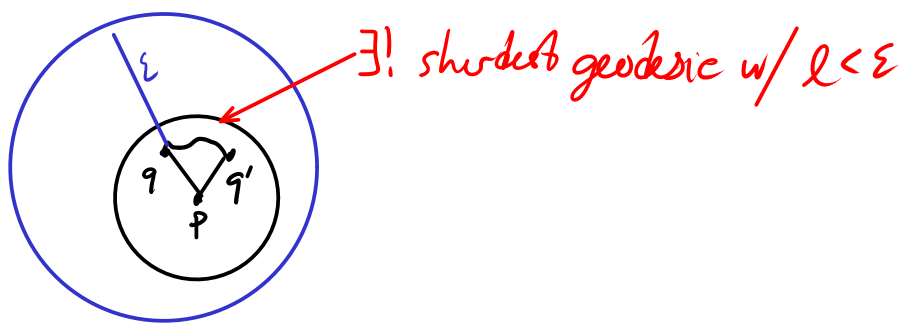
Theorem: For every $p\in M$ there is a neighborhood W and a number $\varepsilon>0$ s.t.

(1) Any $q,q'\in W$ are joined by a unique geodesic of length $< \varepsilon$.

(2) Let $v(q,q')$ denote the unique vector $v \in T_qM$ of length $< \varepsilon$ such that $\exp_q(v) = q'$ [This exists by (1)]. Then $(q,q') \to v(q,q')$ is a smooth map $W \times W \to TM$.

(3) For each $q \in W$, $\exp_q$ maps the open ε-ball in T_qM diffeomorphically onto $U_q \supset W$.

<u>Picture</u>:



<u>Proof</u> Introduce local coordinates $(x^1, \ldots, x^n)$ on M near p.
We get local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ on TM near $(p,0)$,

where $v \in T_qM$ looks like $v = \sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i}$

Recall the neighborhood V of $(p,0)$ on which exp is defined
Define $F : V \to M \times M$
$(q,v) \mapsto (q, \exp_q(v))$

Consider $DF_{(p,0)} : T_{(p,0)}(TM) \to T_{(p,p)}(M \times M)$

Coordinates on $M \times M$ near (p,p) are $(x_1^1, \ldots, x_1^n, x_2^1, \ldots, x_2^n)$
Since $D(\exp_p)_0 = $ identity, we find

Base $DF_{(p,0)}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x_1^i} + \frac{\partial}{\partial x_2^i}$ $\quad$ $[(p,p) \to (p',p')]$

Fiber $DF_{(p,0)}\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x_2^i}$

$\boxed{\text{Thus, } DF_{(p,0)} \text{ is an isomorphism}}$

By the inverse function theorem, F maps some neighborhood V' of $(p,0)$ diffeomorphically onto a neighborhood of $(p,p) \in M \times M$

V' contains a smaller open set V'' of the form
$$V'' = \{(q,v) \mid q \in U, \|v\| < \varepsilon\} \quad \text{where } p \in \underset{\text{open.}}{\underset{\uparrow}{U}} \subset M$$

Let W be an open set $p \in W \subset M$ such that $F(V'') \supset W \times W$.

Let's check that (1), (2), (3) are satisfied.

(1) Take $(q,q') \in W \times W$. Then $F^{-1}(q,q') = (q,v)$ where v satisfies
- $\|v\| < \varepsilon$
- $\exp_q(v) = q'$

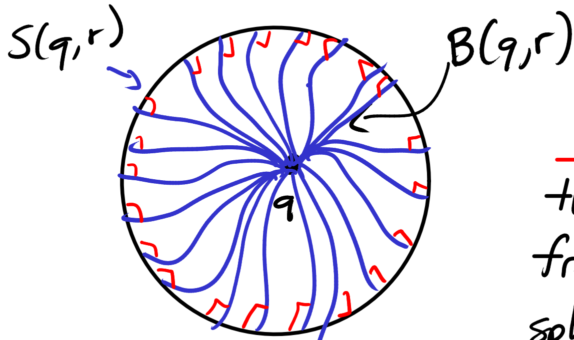
Thus q and q' are joined by a geodesic $\exp_q(tv)$ of length $\|v\| < \varepsilon$
To see uniqueness, suppose $\exists\ w$ of length $< \varepsilon$ such that $\exp_q(w) = q'$
Then $(q,w) \in V''$ and $F(q,w) = F(q,v)$. Since F is a diffeo on V'', $v = w$.

(2) The map in question is just F^{-1}, which is smooth by the inverse function theorem.

(3) For fixed q, the set $\{\exp_q(v) \mid \|v\| < \varepsilon\}$ certainly contains W by (1)
Also for fixed q, F maps $\{q\} \times \{v \in T_qM \mid \|v\| < \varepsilon\}$ to $\{q\} \times \{\exp_q(v) \mid \|v\| < \varepsilon\}$
since F is a diffeo, we're done.

$\blacksquare$

$B(q,r) = \{\exp_q(v) \mid \|v\|_g < r\}$ is called the **geodesic ball with center q and radius r.** The set $S(q,r) = \{\exp_q(v) \mid \|v\| = r\}$ is called the **geodesic sphere.**



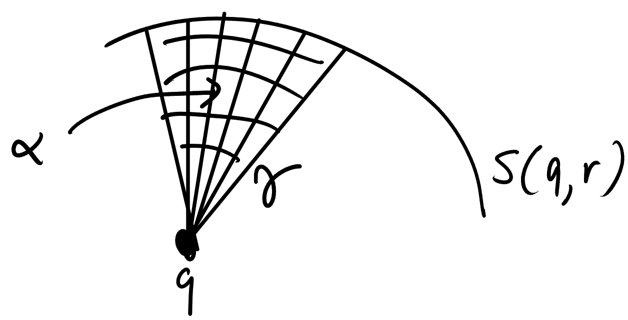
GAUSS' Lemma: For small values of the radius r, the geodesics emanating from q are perpendicular to the geodesic spheres $S(q,r)$.

Proof: Choose $\varepsilon > 0$ and W as before.

Let $v : \mathbb{R} \to T_qM$ be any path with $\|v(s)\| = r < \varepsilon$
$s \mapsto v(s)$ (constant length).

Then $\alpha(s,t) = \exp_q(t \cdot v(s))$ is a parametrized surface, which is a variation of the geodesic $\gamma(t) = \alpha(0,t) = \exp_q(tv(0))$.

Now $\alpha(s,1) \in S(q,r)$.



Note that we can pick α so that $\gamma = \alpha(0,t)$ is any geodesic emanating from q, and so that $\frac{dv}{dt}(0)$ is any tangent vector to $S(q,r)$.

Consider energy

$$E(s) = E(\alpha(s,-)) = \int_0^1 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 dt$$

By the variational formula

$$\frac{d}{ds} E(s) \Big|_{s=0} = -2\int_0^1 g\left(\frac{\partial \alpha}{\partial s}(0,t), \frac{D}{dt}\left(\frac{d\gamma}{dt}\right)\right) dt$$

$$+ g\left(\frac{\partial \alpha}{\partial s}(0,1), \frac{d\gamma}{dt}(1)\right) - g\left(\frac{\partial \alpha}{\partial s}(0,0), \frac{d\gamma}{dt}(0)\right)$$

$$= g\left(\frac{\partial \alpha}{\partial s}(0,1), \frac{d\gamma}{dt}(1)\right)$$

- The integral vanishes because γ is a geodesic.
- Also $\alpha(s,0)=q$ so $\frac{\partial\alpha}{\partial s}(0,0)=0$.

On the other hand $E(s) = \int_0^1 \left\| \frac{\partial\alpha}{\partial t}(s,t)\right\|^2 dt = \int_0^1 \left\| \frac{\partial}{\partial t}\exp(t v(s))\right\|^2 dt$

$$= \int_0^1 \|v(s)\|^2 dt = \int_0^1 r^2\, dt = r^2.$$

So $E(s) = r^2$ is constant and $\frac{dE}{ds}\Big|_{s=0} = 0$!

We conclude

$$g\Big(\frac{\partial\alpha}{\partial s}(0,1), \frac{d\gamma}{dt}(1)\Big) = 0$$

tangent vector to geodesic sphere — tangent vector of geodesic

$\blacksquare$

Corollary: let $c:[a,b] \to U_q = \{ \exp_q(v) \mid \|v\| < \varepsilon \} \setminus \{q\}$

It may be written uniquely as $c(t) = \exp_q(u(t)\cdot v(t))$

where $u:[a,b] \to \mathbb{R}$ satisfies $0 < u(t) < \varepsilon$

and $v(t):[a,b] \to T_qM$ satisfies $\|v(t)\| = 1$

Then

$$\ell(c) \geq |u(b) - u(a)|$$

with equality iff u is monotonic and v is constant.

Proof: set $\alpha(u,t) = \exp_q(u\cdot v(t))$ so $c(t) = \alpha(u(t),t)$

Then $\frac{dc}{dt} = \frac{\partial\alpha}{\partial u}u'(t) + \frac{\partial\alpha}{\partial t}$

$\left\|\frac{\partial\alpha}{\partial u}\right\| = \|v(t)\| = 1$ and $g\Big(\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}\Big) = 0$ by Gauss' lemma.

Thus

$$\left\|\frac{dc}{dt}\right\|^2 = |u'(t)|^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2$$

$$\ell(c) = \int_a^b \left\|\frac{dc}{dt}\right\| dt = \int_a^b \sqrt{|u'(t)|^2 + \left\|\frac{\partial \alpha}{\partial t}\right\|^2}\, dt \geq |u(b) - u(a)|$$

with equality iff $\frac{\partial \alpha}{\partial t} = 0$ and $u'(t)$ always of the same sign,
iff $v(t)$ is constant and u is monotonic. ▨

This corollary says that the paths of minimal length joining $S(q, u(a))$ to $S(q, u(b))$ are the geodesics.

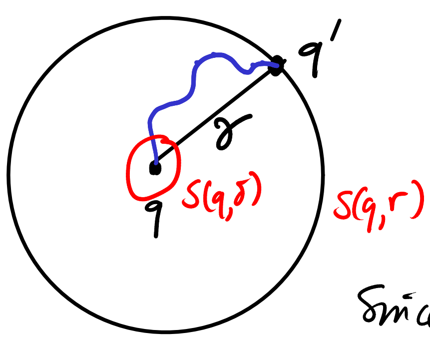
<u>Corollary</u>: Let $\varepsilon > 0$ and W be as before.
Let γ be the geodesic of length $< \varepsilon$ joining q to q'.
Let c be any piecewise smooth path joining q to q'.
Then

$$\ell(\gamma) \leq \ell(c)$$

with equality iff c is a reparametrization of γ.

<u>Proof</u>. We have $q' = \exp_q(rv)$ for $r = \ell(\gamma)$, some v with $\|v\| = 1$.

For any $\delta > 0$, c must connect $S(q,\delta)$ to $S(q,r)$



By the previous corollary

$$\ell(c) \geq r - \delta$$

Since this is true for every δ,

$$\ell(c) \geq r = \ell(\gamma)$$

Since equality can only hold if c is a reparametrization of a geodesic ray between any two spherical shells, we conclude that c would have to be a reparametrization of γ in order for equality to hold. ▨

Finally, we find that sufficiently short geodesics are absolutely length minimizing between their end points