Lecture 13

Geodesics and Exponential map

Now that we know that critical points of $E(\gamma)=\int g(\dot\gamma,\dot\gamma)\,dt$ satisfy $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0$, we study this equation on its own.

<u>Geodesic equation</u>: This is the equation $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0$ for a path $\gamma:[a,b]\to M$. Conceptually, it says that the <u>acceleration</u> of γ is zero, so γ has constant <u>velocity</u> (in the covariant sense). Solutions are geodesics.

Let's express it in local coordinates $(x^1,\dots,x^n)$, $e_i=\frac{\partial}{\partial x^i}$:

$$\gamma(t)=(\gamma^1(t),\dots,\gamma^n(t)) \qquad \frac{d\gamma}{dt}=\sum_{j=1}^n \frac{d\gamma^j}{dt}e_j$$

for $V(t)=\sum_{j=1}^n v^j(t)e_j$ vector field along γ

$$\frac{DV}{dt}=\sum_{k=1}^n\left[\frac{dv^k}{dt}+\sum_{i,j=1}^n \Gamma_{ij}^k\frac{d\gamma^i}{dt}v^j\right]e_k$$

$$\therefore\ \frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=\sum_{k=1}^n\left[\frac{d}{dt}\left(\frac{d\gamma^k}{dt}\right)+\sum_{i,j=1}^n\Gamma_{ij}^k\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\right]e_k$$

This is zero iff each component is zero so the geodesic equation becomes the 2nd order system

$$\boxed{(\forall k=1,\dots,n)\quad \frac{d^2\gamma^k}{dt^2}+\sum_{i,j=1}^n\Gamma_{ij}^k\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}=0}$$

Since the geodesic equation is locally a 2nd order system of ODE with smooth coefficients, the initial value problem is well-posed if we specify the initial point and the initial velocity.

<u>Theorem from ODE theory</u>: For each $p \in M$ and $v \in T_pM$, there is an $\varepsilon > 0$ and a unique smooth path $\gamma : [0,\varepsilon) \to M$ such that

$$\left\{ \begin{array}{l} \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0 \\ \gamma(0) = p \\ \frac{d\gamma}{dt}(0) = v \end{array} \right\}$$

that is, γ is a geodesic starting at p with initial velocity p.

Moreover, γ depends smoothly on the initial data $(p,v) \in TM$ and for any compact subset $K \subset TM$ of initial data, there is a single $\varepsilon > 0$ that works for all $(p,v) \in K$.

<u>Proof</u>: omitted. ☺

<u>Remark</u>: The example $\mathbb{R}^2 - \{(0,0)\}$ shows that the optimal ε may be finite; geodesics do not necessarily exist for all time.

<u>Observation</u>: If $\gamma(t)$ satisfies $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$, and $a \in \mathbb{R}$

then $\tilde{\gamma}(t) = \gamma(at)$ satisfies $\frac{D}{dt}\left(\frac{d\tilde{\gamma}}{dt}\right) = 0$

Indeed $\frac{d\tilde{\gamma}}{dt}(t) = a\frac{d\gamma}{dt}(at)$ and $\frac{D}{dt}\left(a\frac{d\gamma}{dt}(at)\right) = a^2\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)$

(Geodesics are homogeneous with respect to scaling t.)

Exponential map: For $q \in M$ and $v \in T_qM$, let γ be the geodesic such that $\gamma(0) = q$, $\frac{d\gamma}{dt}(0) = v$.

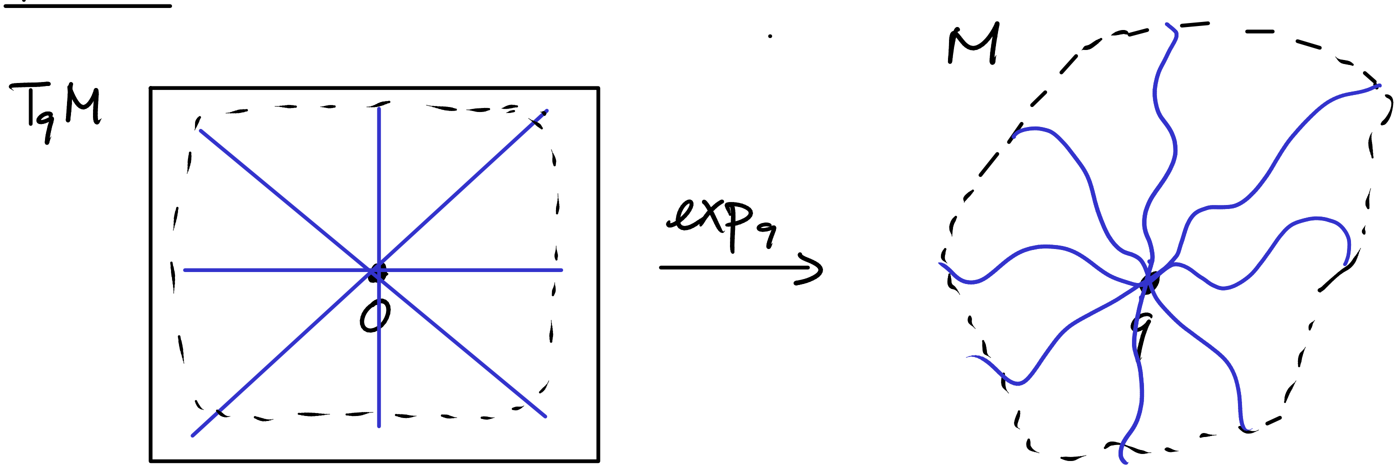
Define $\exp_q(v) := \gamma(1)$ (provided γ can be defined at $t=1$)

This defines a map $\exp_q : \mathcal{U} \to M$, where $\mathcal{U} \subset T_qM$ called the exponential map at $q \in M$.

(1) Note that if $\gamma(1)$ exists, then $\gamma(t) = \exp_q(tv)$ for $t \in [0,1]$, by homogeneity.

(2) By the ODE theorem, the domain $\mathcal{U}$ of $\exp_q$ contains an open set around 0. ($\exists\, \varepsilon > 0$ such that $\forall\, v$ with $\|v\|_g \le \varepsilon$, $\exp_q(v)$ exists.)

Picture:



(3) The geodesics in M passing through q are the images of the straight lines in T_qM under $\exp_q$.