RG III: Covariant derivatives and calculus of variations for energy.

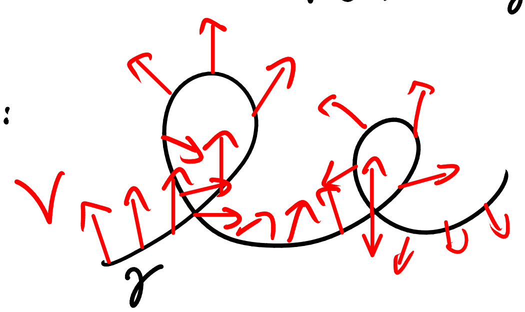
Recall: a Riemannian manifold (M, g) has a connection ∇ on its tangent bundle, called the Levi-Civita connection, that is torsion free and compatible with g.

Today we want to use ∇ to study paths.

Covariant Derivative along a path: Let $\gamma : [a,b] \to M$ be a smooth path. Let $V(t)$ denote a vector field along γ. Thus, for each $t \in [a,b]$, we get a vector

$$V(t) \in T_{\gamma(t)} M$$

Picture:



This is not the same as a vector field on M restricted to γ. Rather, vector fields along γ are sections of the pull back bundle

$$\begin{array}{ccc} \gamma^* TM & \longrightarrow & TM \\ \downarrow & & \downarrow \\ [a,b] & \xrightarrow{\gamma} & M \end{array}$$

We want to differentiate $V(t)$ with respect to the parameter t. Since $V(t+h)$ and $V(t)$ live in different spaces, we need to use the connection ∇ to relate them. This yields an operation we denote

$$\{\text{vector fields along } \gamma\} \longrightarrow \{\text{vector fields along } \gamma\}$$

$$V(t) \longmapsto \frac{DV}{dt}$$

The existence and uniqueness of this operation is the following.

<u>Proposition</u>: There is a unique operation $V \mapsto \frac{DV}{dt}$ such that

(1) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$ (linearity)

(2) $\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt}$ for $f:[a,b]\to\mathbb{R}$ smooth function (Leibniz rule)

(3) If $V(t) = X(\gamma(t))$ for some X a vector field on M defined in a neighborhood of γ, then

$$\frac{DV}{dt} = \nabla_{\dot\gamma} X \quad \left[\text{more pedantically } \frac{DV}{dt}(t_0) = \left(\nabla_{\dot\gamma(t_0)} X\right)(\gamma(t_0))\right]$$

where as always $\dot\gamma = D\gamma\left(\frac{\partial}{\partial t}\right)$ is the velocity.
(this property connects $\frac{D}{dt}$ to ∇)

<u>Proof</u>: We can actually express $\frac{D}{dt}$ locally in terms of Γ_{ij}^k, the Christoffel symbols of ∇.

Choose local coordinates around $p = \gamma(t_0)$. Then for t close to t_0 we can write

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)) \quad \text{components.}$$

Write $e_i = \frac{\partial}{\partial x^i}$ coordinate vector fields.

To emphasize dependence on t, write $e_i(\gamma(t))$ for the vector in $T_{\gamma(t)}M$.

Since $\{e_i(\gamma(t))\}_{i=1}^n$ is a basis of $T_{\gamma(t)}M$ for each γ, any vector field $V(t)$ along γ may be written

$$V(t) = \sum_{i=1}^n v^i(t)\, e_i(\gamma(t)) \qquad (v^i \text{ are functions})$$

Part 1: Uniqueness of $\frac{D}{dt}$: Suppose $\frac{D}{dt}$ exists and satisfies (1), (2), (3)

By (1) and (2) we find: $\frac{DV}{dt} = \frac{D}{dt}\left(\sum_{i=1}^n v^i(t)\, e_i(\gamma(t))\right) =$

$$= \sum_{i=1}^n \frac{dv^i}{dt} e_i(\gamma(t)) + \sum_{i=1}^n v^i(t) \frac{D(e_i(\gamma(t)))}{dt}$$

We can apply (3) to determine $\frac{D}{dt}(e_i(\gamma(t)))$.

It equals $\nabla_{\dot\gamma} e_i = \nabla_{\sum_{j=1}^n \frac{d\gamma^j}{dt} e_j} e_i = \sum_{j=1}^n \frac{d\gamma^j}{dt} \nabla_{e_j} e_i$

$$= \sum_{j=1}^n \frac{d\gamma^j}{dt} \sum_{k=1}^n \Gamma_{ji}^k e_k$$

We find

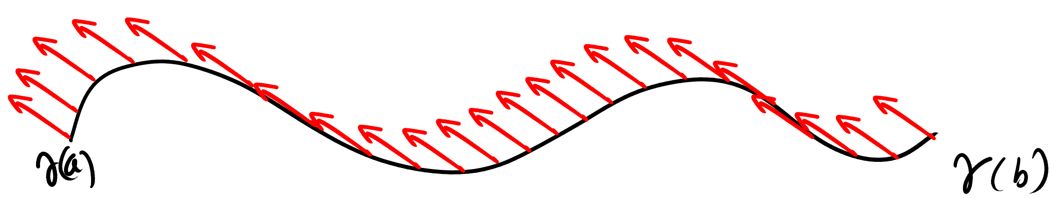
$$\frac{DV}{dt}(t_0) = \sum_{k=1}^n \left[\frac{dv^k}{dt}(t_0) + \sum_{i,j=1}^n \Gamma_{ij}^k(\gamma(t_0)) \frac{d\gamma^i}{dt}(t_0)\, v^j(t_0)\right] e_k(\gamma(t_0))$$

or readably $$\frac{DV}{dt} = \sum_k \left[\frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{dt} v^j\right] e_k$$

For existence of $\frac{D}{dt}$, we just take this formula and show that it satisfies (1), (2), (3). $\blacksquare$

Remarks: ① Just as vector fields along γ are sections of the pullback bundle $\gamma^*TM \to [a,b]$, the operator $\frac{D}{dt}$ is really $(\gamma^*\nabla)_{\frac{\partial}{\partial t}}$ where $\gamma^*\nabla$ is the pullback connection.

(2) Parallel transport of vectors can be expressed in terms of $\frac{D}{dt}$.
Definition: a vector field $V(t)$ along a path $\gamma : [a,b] \to M$ is parallel if $\frac{DV}{dt} \equiv 0$. When this holds, $V(b)$ is the parallel transport of $V(a)$ along γ.



[This is consistent with our previous notion of parallel transport]

Last week we said we want to find paths that minimize length

$$\ell(\gamma) = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt$$

It is more convenient to minimize the Energy (aka Action)

$$E(\gamma) = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))\, dt$$

There is a close relationship between the minimizers of these two functionals, which we will discuss eventually.

Calculus of variations for $E(\gamma)$.

Pick points $p, q \in M$ and consider the space of paths from p to q

$$\Omega_{p,q}M = \{ \gamma : [0,1] \to M \mid \gamma \text{ smooth}, \gamma(0)=p, \gamma(1)=q \}$$

The energy is a function:

$$E : \Omega_{p,q}M \longrightarrow \mathbb{R}$$

And we want to find the minima of E.

Key Ideas of Calculus of variations:

(1) The path space is "morally" an infinite dimensional manifold.

(2) To find the minima of a function, calculate the differential dE and see where it's zero.

(3) A tangent vector to $\Omega_{p,q}M$ at a point γ is a vector field along γ.

$$\text{"}V \in T_\gamma(\Omega_{p,q}M)\text{"} \longleftrightarrow V(t) \in T_{\gamma(t)}M$$

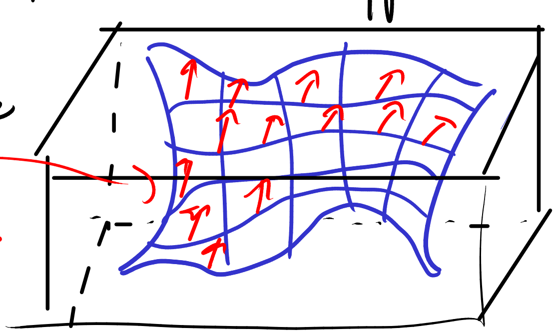
(4) A path in $\Omega_{p,q}M$ is a parametrised surface in M.

$$\begin{pmatrix} \alpha : (-\varepsilon, \varepsilon) \to \Omega_{p,q}M \\ \alpha(s) \in \Omega_{p,q}M \end{pmatrix} \longleftrightarrow \begin{pmatrix} \alpha : (-\varepsilon,\varepsilon) \times [0,1] \longrightarrow M \\ \alpha(s,t) \in M \end{pmatrix}$$

To make things rigorous, we focus on the last point (4), since it doesn't make reference to a supposed manifold structure on $\Omega_{p,q}M$.

Parametrised surface

Vector field along surface.



Let $\alpha(s,t)$ be a parametrized surface in M.
Then $\frac{\partial\alpha}{\partial s} = D\alpha\left(\frac{\partial}{\partial s}\right)$ and $\frac{\partial \alpha}{\partial t} = D\alpha\left(\frac{\partial}{\partial t}\right)$
are vector fields along the parametrized surface α.

For vector fields along a parametrized surface $V(s,t)$, define

$\frac{DV}{\partial s}(s_0,t_0) =$ fix $t=t_0$, and take covariant derivative of $V(s,t_0)$ along the path $s \mapsto \alpha(s,t_0)$

$\frac{DV}{\partial t}(s_0,t_0) =$ fix $s=s_0$, and take covariant derivative of $V(s_0,t)$ along the path $t \mapsto \alpha(s_0,t)$

<u>Lemma</u>: If $\alpha(s,t)$ is a parametrized surface in M, and ∇ is any torsion free connection in TM (in particular if ∇ is the Levi-Civita connection) then

$$\frac{D}{\partial s}\left(\frac{\partial\alpha}{\partial t}\right) = \frac{D}{\partial t}\left(\frac{\partial \alpha}{\partial s}\right)$$

<u>Proof</u> Homework.

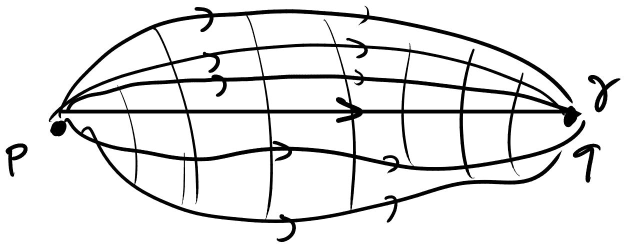
<u>Now for calculus of variations</u>: Suppose that $\gamma \in \Omega_{p,q}M$ is a local minimum of $E: \Omega_{p,q}M \to \mathbb{R}$. (We don't know if such γ exist, but press on)
Consider any path in $\alpha: (-\varepsilon,\varepsilon) \to \Omega_{p,q}M$

$$s \longmapsto \alpha(s)$$

such that $\alpha(0)=\gamma$. We think of α as a parametrized surface

$$\alpha : (-\varepsilon,\varepsilon)\times[0,1] \to M$$
$$(s, t) \longmapsto \alpha(s,t)$$

such that $\alpha(0,t)=\gamma(t)$, $\alpha(s,0)=p$, $\alpha(s,1)=q$



Assume α is smooth.

Since γ is a local minimum, we know $E(\alpha(s)) \geq E(\alpha(0)) = E(\gamma)$ for all $s \in (-\varepsilon, \varepsilon)$. But this immediately implies

$$\frac{d}{ds}\big(E(\alpha(s))\big)\Big|_{s=0} = 0$$

We are going to calculate $\frac{d}{ds}\left(E(\alpha(s))\right)$

$$E(\alpha(s)) = \int_0^1 g\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\right) dt$$

$$\frac{d}{ds} E(\alpha(s)) = \int_0^1 \frac{\partial}{\partial s} g\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\right) dt$$

Now it follows from the fact that ∇ is compatible with g that

$$\frac{\partial}{\partial s} g\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\right) = g\left(\frac{D}{\partial s}\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\right) + g\left(\frac{\partial \alpha}{\partial t}, \frac{D}{\partial s}\frac{\partial \alpha}{\partial t}\right)$$

$$= 2g\left(\frac{D}{\partial s}\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\right)$$

So $\displaystyle\frac{d}{ds} E(\alpha(s)) = 2\int_0^1 g\left(\frac{D}{\partial s}\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}\right) dt$

$$= 2\int_0^1 g\left(\frac{D}{\partial t}\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right) dt \qquad \text{by torsion free lemma.}$$

Now $\displaystyle\frac{\partial}{\partial t} g\left(\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right) = g\left(\frac{D}{\partial t}\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right) + g\left(\frac{\partial \alpha}{\partial s}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t}\right)$

This lets us do "integration by parts"

$$\frac{d}{ds} E(\alpha(s)) = 2\, g\left(\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right)\Big|_{t=0}^{t=1} - 2\int_0^1 g\left(\frac{\partial \alpha}{\partial s}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t}\right) dt$$

Since $\alpha(s,0) = p$ and $\alpha(s,1) = q$, we have $\frac{\partial \alpha}{\partial s}(s,0) = 0$, $\frac{\partial \alpha}{\partial s}(s,1) = 0$ $\Rightarrow g\left(\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right)\Big|_0^1 = 0$

So $\frac{d}{ds} E(\alpha(s)) = -2\int_0^1 g\left(\frac{\partial \alpha}{\partial s}, \frac{D}{\partial t}\frac{\partial \alpha}{\partial t}\right) dt$

Set $s=0$. $\alpha(0,t) = \gamma(t)$, so $\frac{\partial \alpha}{\partial s}(0,t) =: V(t)$

and $\frac{\partial \alpha}{\partial t}(0,t) = \frac{d\gamma}{dt}(t) = \dot{\gamma}(t)$ are vector fields along γ.

and $\frac{d}{ds} E(\alpha(s))\Big|_{s=0} = -2\int_0^1 g\left(V(t), \frac{D}{dt}\dot{\gamma}\right) dt$

Since this must be zero no matter what α is, it must be zero no matter what V is, which means that $\frac{D}{dt}\dot{\gamma} \equiv 0$.

Interpretation: (1) $\frac{\partial \alpha}{\partial t} = \frac{d\gamma}{dt} = \dot{\gamma}$ is the velocity of γ.

(2) $\frac{\partial \alpha}{\partial s} = V(t)$ is the infinitesimal variation of γ.

(3) $\frac{D}{dt}\dot{\gamma} = A(t)$ is the covariant acceleration of γ.

(4) the critical points of E are the paths with zero covariant acceleration. We give them a name.

Def: A path γ is called a geodesic if $\frac{D}{dt}\dot{\gamma} = 0$.