Riemannian Geometry II: Levi-Civita connection.

Now we can reap the fruits of our labor in setting up the theory of connections. For any Riemannian manifold (M,g), there is a canonical connection in the tangent bundle TM, which is determined uniquely by the metric g.

Let (M,g) be a Riemannian manifold. We consider connections in the tangent bundle TM, though of as a covariant derivative ∇. Recall that $\mathfrak{X}(M) = \Gamma(M, TM)$ is the set of vector fields on M. Thus a connection in TM is a map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

Satisfying the properties we set down before.

(i) $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z \qquad (a,b \in \mathbb{R})$

(ii) $\nabla_X(fY) = (X.f)Y + f\nabla_X Y \qquad (f \in C^\infty(M))$

(iii) $\nabla_{(fX+gZ)} Y = f\nabla_X Y + g\nabla_Z Y \qquad (f,g \in C^\infty(M))$

The neat thing about connections in TM (as opposed to connections in a general vector bundle E) is that the space of sections of TM <u>is</u> the space of vector fields, so ∇ is letting us differentiate a vector field Y with respect to another vector field X, and the result is a third vector field $\nabla_X Y$.

How could ∇ be related to the metric g?

Let's go back to $\mathbb{R}^n$. Let $\vec{v}$ and $\vec{w}$ be vector fields on $\mathbb{R}^n$, thought of as functions $\vec{v}, \vec{w} : \mathbb{R}^n \to \mathbb{R}^n$. It is certainly true that

$$\underbrace{\frac{\partial}{\partial x^i}\left(\vec{v}\cdot\vec{w}\right)}_{\text{This is the derivative of a function.}} = \frac{\partial \vec{v}}{\partial x^i}\cdot\vec{w} + \vec{v}\cdot\frac{\partial \vec{w}}{\partial x^i}$$

These are derivatives of vector fields

We would like to generalize this to a Riemannian manifold (M,g)

Standard $\mathbb{R}^n$		(M,g)
$\vec{v}, \vec{w}$	$\longleftrightarrow$	$X, Y \in \mathfrak{X}(M)$
dot product $\bullet$	$\longleftrightarrow$	metric g
$\left\{ \text{derivative of function } \dfrac{\partial f}{\partial x^i} \right\}$	$\longleftrightarrow$	$\left\{ \text{action of vector field } Z \text{ on function } f : Z.f \right\}$
$\left\{ \text{derivative of vector field } \dfrac{\partial \vec{v}}{\partial x^i} \right\}$	$\longleftrightarrow$	$\left\{ \text{covariant derivative } \nabla_Z X \text{ of } X \text{ in direction of } Z. \right\}$

Thus, the identity we want is written

$$\circledast \quad Z.\big(g(X,Y)\big) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Not every connection ∇ satisfies this! If g is given, this is a condition on ∇.

Def A connection ∇ in TM is compatible with g if the identity ⊛ is satisfied by all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Remark: this definition makes sense for a connection in any vector bundle equipped with a Euclidean metric.

For connections in the tangent bundle TM, there is another special property we can consider: Recall the definition of Lie bracket, in $\mathbb{R}^n$ or in local coordinates $(x^1, \dots, x^n)$ on $U \subset M$.

Let $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$ be vector fields

The Lie bracket $[X,Y]$ is the vector field

$$[X,Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

These terms "look like" derivaties of vector fields. Are they the same as the corresponding covariant derivatives ?

Translating in terms of ∇, what we are asking is

$$(**) \qquad [X,Y] = \nabla_X Y - \nabla_Y X \quad ?$$

Def The function $T: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is the torsion of ∇. The connection ∇ is torsion free if $T=0$, which is the same as saying (**) holds.

Remark: for any connection ∇ in TM, the torsion T is tensorial, i.e. $C^\infty(M)$-linear in each factor. Therefore, by Homework 3 Problem 1, T corresponds to a section of $(T^*M)\otimes(T^*M)\otimes TM$, and we can write

$$T = \sum_{i,j,k=1}^{n} T_{ij}^{k}\, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \quad \text{(in local coordinates)}$$

Theorem Let (M,g) be a Riemannian manifold. There is a unique connection ∇ in TM that is compatible with g and torsion free. This ∇ is called the Levi-Civita connection.

Unfortunately no proof of this theorem seems to be very intuitive.

Proof. We first prove uniqueness by getting a formula for ∇ in terms of g and intrinsic operations on M. Then we check that this formula actually has the right properties.

Suppose that ∇ is compatible with g and torsion free. Let X, Y, Z be vector fields on M. By compatibility with g:

$$\text{(I)} \quad X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$\text{(II)} \quad Y.g(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$\text{(III)} \quad Z.g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Now consider (I) + (II) − (III)

$$X \cdot g(Y,Z) + Y \cdot g(Z,X) - Z \cdot g(X,Y)$$

$$= \left\{ \begin{array}{l} {}^{1}g(\nabla_X Y, Z) + {}^{2}g(Y, \nabla_X Z) \\ + {}^{3}g(\nabla_Y Z, X) + {}^{4}g(Z, \nabla_Y X) \\ - {}^{5}g(\nabla_Z X, Y) - {}^{6}g(X, \nabla_Z Y) \end{array} \right\}$$

1 2 3 4 5 6 just to keep track of terms

$$= \left\{ \begin{array}{l} {}^{1}g(\nabla_X Y, Z) + {}^{4}g(\nabla_Y X, Z) \\ + {}^{2}g(\nabla_X Z, Y) - {}^{5}g(\nabla_Z X, Y) \\ + {}^{3}g(\nabla_Y Z, X) - {}^{6}g(\nabla_Z Y, X) \end{array} \right\}$$

Rearrange terms and use that g is symmetric

$$= \left\{ \begin{array}{l} g(\nabla_X Y + \nabla_Y X, Z) \\ + g(\nabla_X Z - \nabla_Z X, Y) \\ + g(\nabla_Y Z - \nabla_Z Y, X) \end{array} \right\}$$

by bilinearity of g

$$= \left\{ \begin{array}{l} g(\nabla_X Y + \nabla_X Y + [Y,X], Z) \\ + g([X,Z], Y) \\ + g([Y,Z], X) \end{array} \right\}$$

By torsion freeness e.g. $\nabla_Y X = \nabla_X Y + [Y,X]$

Now there is only one term that involves ∇:

$$X.\,g(Y,Z) + Y.\,g(Z,X) - Z.\,g(X,Y)$$
$$= 2g(\nabla_X Y, Z) + g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X)$$

This shows that $g(\nabla_X Y, Z)$ can be solved for in terms of the metric g, the Lie bracket $[\,,\,]$, and the action of vector fields on functions. We can now conclude that ∇ is unique, if it exists. For suppose that ∇^1 and ∇^2 are two such connections. Then

$$g(\nabla^1_X Y, Z) = g(\nabla^2_X Y, Z) \quad \text{for all } X,Y,Z \in \mathfrak{X}(M)$$

Then $g(\nabla^2_X Y - \nabla^1_X Y, Z) = 0$ for all $X,Y,Z \in \mathfrak{X}(M)$

Since g is nondegenerate, the only vector field that is orthogonal to all vector fields Z is 0. Hence $\nabla^2_X Y - \nabla^1_X Y = 0$.

For existence, the idea is to actually define ∇ by the relation

$$2g(\nabla_X Y, Z) = X.\,g(Y,Z) + Y.\,g(Z,X) - Z.\,g(X,Y) - \big[g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X)\big]$$

<u>Sketch</u>: First, show that, for fixed X and Y, the right hand side is tensorial in the Z variable. Thus the right hand side defines a section of T^*M for fixed X and Y.

Since g is nondegenerate, any section $\alpha \in \Gamma(M, T^*M)$ is of the form $\alpha = g(A, -)$ for some section $A \in \Gamma(M, T^*M)$

So for fixed X and Y, there is a vector field A such that $g(A, Z) =$ Right hand side

If we define $\nabla_X Y = \frac{1}{2} A$, then the defining relation is satisfied. Now we need to check that $\nabla_X Y$ actually defines a connection: (1) linear in Y, (2) Leibniz rule, (3) tensorial in X. These follow from direct manipulation of the defining relation. $\blacksquare$

Now that we know that the Levi-Civita exists, we can compute it in local coordinates $(x^1, \dots, x^n)$. Let $\frac{\partial}{\partial x^i}$ be the coordinate vector fields. They are also a frame for TM, so the connection is determined by the Christoffel symbols

$$\Gamma_{ij}^k \text{ such that } \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

The defining relation for ∇ drastically simplifies when X, Y, Z are coordinate vector fields $\frac{\partial}{\partial x^i}$, since the Lie bracket terms vanish:

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$$

$$2g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + \frac{\partial}{\partial x^j} g\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right) - \frac{\partial}{\partial x^k} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

Since $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$, we get

$$2g\left(\sum_l \Gamma_{ij}^l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) = \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

$$\sum_l 2 g_{lk} \Gamma_{ij}^l = \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

$$\Gamma_{ij}^l = \sum_k \frac{1}{2} g^{lk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

where (g^{lk}) is the inverse matrix of (g_{lk})

That is: $\sum_k g_{ik} g^{jk} = \delta_i^j$

Remark The fact that ∇ is torsion free boils down to

$$\Gamma_{ij}^k = \Gamma_{ji}^k \quad (\Gamma \text{ is symmetric in } i \text{ and } j)$$