Lecture 10

Start Riemannian Geometry.

We specialize the theory of bundles and connections to a case of principal importance: The tangent bundle TM of a smooth manifold M.

Recall that a Euclidean metric g on a vector space V is a symmetric positive-definite bilinear form on V:

$$g : V \times V \to \mathbb{R} \quad \left(\text{or } g : V \otimes V \to \mathbb{R} \quad \text{or } g \in V^\vee \otimes V^\vee \right)$$

such that $g(u,v) = g(v,u)$, $g(v,v) \geq 0$ and $g(v,v) = 0$ iff $v = 0$.

A Euclidean metric on a vector bundle $\pi : E \to M$ is such a structure on each fiber $E_p = \pi^{-1}(p)$ varying smoothly with $p \in M$. $(g \in \Gamma(M, T^*M^{\otimes 2}))$

Def Let M be a smooth manifold. A Riemannian metric on M is a Euclidean metric on the tangent bundle $TM \to M$. A Riemannian manifold is a pair (M, g) where g is a Riemannian metric on M.

Thus, in a Riemannian manifold, each tangent space T_pM is a Euclidean space with inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. In other words, M is "infinitesimally Euclidean", but AT LARGE SCALES M IS NONEUCLIDEAN as we shall see.

"Un espace de Riemann est au fond formé d'une infinité de petits morceaux d'espaces euclidiens." E. Cartan

What do we get from a Riemannian metric?

Lengths of tangent vectors: If $p \in M$, and $v \in T_pM$, and g_p denotes the metric on T_pM, then $g_p(v,v)$ may be interpreted as the squared length of v, just as $\langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2$ in $\mathbb{R}^n$

$$\text{length of } v \text{ w.r.t. } g = \|v\|_g := \sqrt{g_p(v,v)}$$

Angles: For $v, w \in T_pM$ define the angle $\angle(v,w)$ by

$$\cos(\angle(v,w)) = \frac{g(v,w)}{\|v\|_g \cdot \|w\|_g}$$

Orthogonality: $v \perp w \iff g(v,w) = 0$

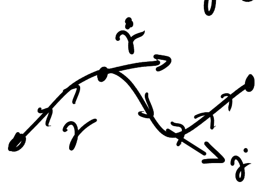
All these notions are about a single tangent space. More interesting is

The length of a differentiable path: Let $\gamma : [a,b] \to M$ be a smooth path (at least continuous first derivative)

We denote by $\dot{\gamma}$ the velocity vector of γ

$$\dot{\gamma}(t_0) = \left.\frac{d\gamma}{dt}\right|_{t_0} = D\gamma_{t_0}\left(\frac{\partial}{\partial t}\right)$$

coordinate vector field on $[a,b] \subset \mathbb{R}$

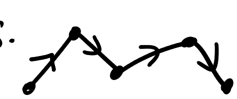


The length of γ is

$$\ell(\gamma) := \int_a^b \|\dot{\gamma}\|_g \, dt = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt$$

Observe that $\ell(\gamma) \geq 0$ and $\ell(\gamma) = 0$ iff γ is a constant path.
(since g is positive definite)

We can also measure lengths of piecewise smooth paths.

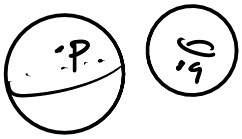


Riemannian distance function: Let $p,q \in M$ be points. Define the distance

$$\mathrm{dist}_g(p,q) = \inf \left\{ \ell(\gamma) \;\middle|\; \begin{array}{l} \gamma : [a,b] \to M \quad \gamma(a)=p,\ \gamma(b)=q \\ \gamma \text{ is continuous and piecewise smooth} \end{array} \right\}$$

= the infimum of all lengths of paths joining p to q.

Note that $\mathrm{dist}_g(p,q) = \infty$ if p and q cannot be joined by a path

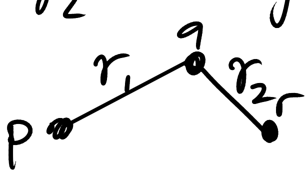


But if M is connected (in the topological sense) any $p,q \in M$ may be connected by a piecewise smooth path, so $\mathrm{dist}_g(p,q) < \infty$.

<u>Prop</u> If M is connected, then $\mathrm{dist}_g : M \times M \to \mathbb{R}$ is a metric-space distance function, that is:

(a) $\mathrm{dist}_g(p,q) = \mathrm{dist}_g(q,p)$ (symmetric)
(b) $\mathrm{dist}_g(p,r) \leq \mathrm{dist}_g(p,q) + \mathrm{dist}_g(q,r)$ (triangle ineq)
(c) $\mathrm{dist}_g(p,q) \geq 0$ and $= 0$ iff $p = q$. (positive definite)

<u>Proof</u> (a) follows by reversing peramatrization of paths.
For (b), consider that if γ_1 is a piecewise smooth path joining p to q and γ_2 is one joining q to r, then the concatenation $\gamma_1 \# \gamma_2$ is a path joining p to r, and



$$\ell(\gamma_1 \# \gamma_2) = \ell(\gamma_1) + \ell(\gamma_2)$$

Thus $\mathrm{dist}_g(p,r) \leq \ell(\gamma_1) + \ell(\gamma_2)$ for any γ_1 and γ_2

Now consider a sequence $\gamma_{1,n}$ joining p to q such that
$$\lim_{n\to\infty} \ell(\gamma_{1,n}) = \mathrm{dist}_g(p,q)$$

For any γ_2, $\text{dist}_g(p,r) \le \ell(\gamma_{1,n}) + \ell(\gamma_2)$

Take limit $n\to\infty$:

$$\text{dist}_g(p,r) \le \text{dist}_g(p,q) + \ell(\gamma_2)$$

Now take $\gamma_{2,n}$ such that $\lim_{n\to\infty} \ell(\gamma_{2,n}) = \text{dist}_g(q,r)$

$$\text{dist}_g(p,r) \le \text{dist}_g(p,q) + \ell(\gamma_{2,n})$$

Take limit: $\text{dist}_g(p,r) \le \text{dist}_g(p,q) + \text{dist}_g(q,r)$

(3) follows from the observation at the bottom of page 2 above. $\blacksquare$

Thus any Riemannian manifold becomes a metric space. It turns out that the topology induced from dist_g is the same as the one we started with, but we'll prove this later.

When we see something like

$$\text{dist}_g(p,q) = \inf\left\{ \ell(\gamma) \;\middle|\; \gamma \text{ is a path joining } p \text{ to } q \right\}$$

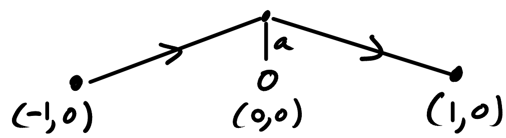
the natural question is Is there a minimizer? Is there a path γ joining p to q such that $\text{dist}_g(p,q) = \ell(\gamma)$?
That would be great, because the shortest path between two points would be the analogue of a "straight line".

The general answer is NO. Counterexample:
Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $p = (-1,0)$ and $q = (1,0)$
Let g = Standard Euclidean metric in $\mathbb{R}^2$, restricted to M.
It is easy to see that lengths in the Riemannian sense agree with the usual notion of length from multivariable calculus.

Since any path in M is also a path in $\mathbb{R}^2$, we see that

$$\operatorname{dist}_{(M,g)}\big((-1,0),(1,0)\big) \geq \operatorname{dist}_{\mathbb{R}^2}\big((-1,0),(1,0)\big) = 2$$

But there are paths in M of length $2+\varepsilon$ for every $\varepsilon > 0$. Consider



which has length $2\sqrt{1+a^2}$. As $a \to 0$, this length $\to 2$.
We conclude $\operatorname{dist}_{(M,g)}\big((-1,0),(1,0)\big) = 2$

But there is no path in M of length 2 joining $(-1,0)$ to $(1,0)$.
If γ were such a path, γ also lies in $\mathbb{R}^2$, and the only shortest path in $\mathbb{R}^2$ joining $(-1,0)$ to $(1,0)$ is the line segment through to origin.

To get around this difficulty and obtain a definition of "straight lines" in M (more properly called geodesics), there are several non obvious ideas that we must develop.

(1) Replace the length functional $\ell(\gamma)$ by the Energy functional

$$E(\gamma) = \int_a^b \|\dot{\gamma}\|_g^2 \, dt = \int_a^b g_{\gamma(t)}\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)\, dt$$

Minimizers of $E(\gamma)$ also minimize $\ell(\gamma)$, and the converse holds up to reparametrization. At any rate $E(\gamma)$ is easier to analyze because it doesn't have a square root in it.

(2) The condition that γ minimizes $E(\gamma)$ may be translated into a second-order differential equation satisfied by γ. This translation process is called *the calculus of variations*, and it leads to the *geodesic equation*.

(3) Unfortunately, calculus of variations still cannot prove the existence of minimizers, so we study the geodesic equation on its own. It turns out that the geodesic equation can be conveniently expressed via a connection in the tangent bundle $TM \to M$, *the Levi-Civita connection*.

(4) With all this in hand, we will show that M contains a wealth of geodesics, which are locally length minimizing paths. The existence of minimizing paths with fixed endpoints is an extra condition related to completeness of M as a metric space.

<u>Next time</u>: Levi-Civita connection.