Differentiable manifolds II Lecture 1

This course will cover various topics in the theory of manifolds that revolve around the central notion of <u>curvature</u>.

The curvature of a Riemannian manifold is a fundamental quantity which distinguishes, for example, Euclidean, Spherical, and Hyperbolic geometries.

Riemannian curvature is a special case of the curvature of a connection in a vector bundle, so we will study vector bundles first.

In fact, this more general notion of curvature is also widely used, for example in the "Gauge theories" of elementary particle physics. (and the mathematics deriving from it)

We will also study the relationship between curvature and topology, and the theory of characteristic classes.

<u>Vector Bundles</u>: Since we are starting with this rather abstract notion, we should provide some motivation:

Let X be a manifold.

<u>Question</u>: What is a vector-valued function on X?

Obviously, the answer is a continuous, or better, smooth function $F: X \longrightarrow \mathbb{R}^k$

Strictly speaking, this is the correct answer, but philosophically, this concept is deficient.

A manifold is built up from pieces of Euclidean space (charts). As X looks locally like Euclidean space we could also consider objects which "look locally like vector-valued functions on Euclidean space"!

You ought to be wondering "how can that even be a different thing?" The idea is that the vector space $\mathbb{R}^k$ where the "function" takes values is not absolute, but relative to the point $x \in X$ where we evaluate the function.

<u>Very informal definition</u>: Let X be a manifold, and let k be a positive integer.
A <u>rank-k vector bundle</u> over X consists of a <u>assignment</u> $x \longmapsto E_x$ of a k-dimensional $\mathbb{R}$-vector space E_x to each point $x \in X$.

The disjoint union of the vector space E_x is denoted E:

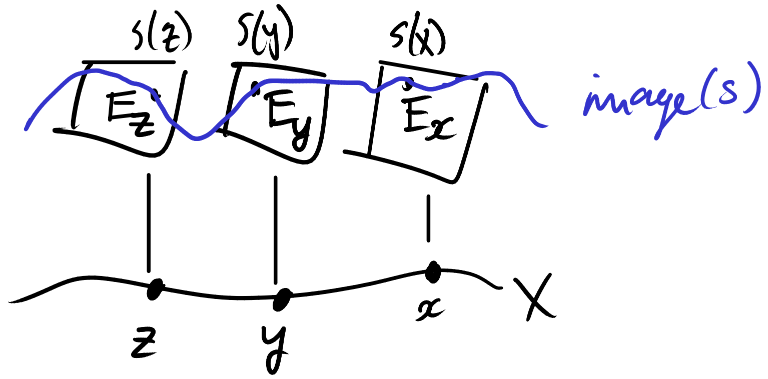
$$E = \coprod_{x \in X} E_x$$

(Recall disjoint union of sets A_i indexed by $i \in I$ may be defined as the (plain) union $\bigcup_{i \in I} \{i\} \times A_i$ when A_i not pairwise disjoint.)

A <u>section</u> of the vector bundle is a function

$$s : X \to E$$

such that $s(x) \in E_x \subset E$



The original term for section was "cross-section".

This is an informal definition because there are no conditions of differentiability, or even continuity.

<u>Definition</u>: Let S be a set. An <u>$\mathbb{R}$-vector space structure</u> on S is a pair of maps $+ : S \times S \to S$, $\bullet : \mathbb{R} \times S \to S$ satisfying the axioms of a vector space with $+$ as vector addition and $\bullet$ as scalar mult. It is called <u>k-dimensional</u> if $(S, +, \bullet)$ is spanned by k linearly independent vectors. (Usual definition)

<u>Definition</u>: A rank-k <u>vector bundle</u> over X consists of a topological space E and a continuous map $\pi : E \to X$ and for each $x \in X$, the structure of a k-dimensional $\mathbb{R}$-vector space on $\pi^{-1}(x) =: E_x$
(Thus there are operations $+_x : E_x \times E_x \to E_x$, $\underset{x}{\bullet} : \mathbb{R} \times E_x \to E_x$)

The following <u>local triviality condition</u> must be satisfied

<u>local triviality</u>:

$\forall x \in X$, $\exists\, \mathcal{U}$ open set, $x \in \mathcal{U}$ and $\exists\, \phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$ such that

(1) $\phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$ is a homeomorphism

(2) The diagram

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}) & \xrightarrow{\phi} & \mathcal{U} \times \mathbb{R}^k \\ \pi \downarrow & & \downarrow pr_1 \\ \mathcal{U} & = & \mathcal{U} \end{array}$$

commutes.

(3) $\phi|_{\pi^{-1}(x)} : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^k$ is an isomorphism of vector spaces.

Note that (1) and (2) imply that $\phi|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^k$ is a homeomorphism, so (3) just says that the vector space operations are respected by our local trivialization. Note that $\pi^{-1}(x)$ was assumed to have a vector space structure, and $\{x\} \times \mathbb{R}^k$ is obviously in bijection with $\mathbb{R}^k$ which has the canonical vector space structure.

The above definition does not use the fact that X is a differentiable manifold. In fact it is the definition of a topological vector bundle over a topological space X.

<u>Definition</u> A <u>C^∞-differentiable vector bundle</u> over a C^∞-differentiable manifold X is $\pi : E \to X$ as above, where additionally E is provided with the structure of a C^∞-manifold, π is a C^∞-map, and the maps ϕ in the local triviality condition may be taken to be C^∞-maps rather than merely homeomorphisms.

Def A <u>section</u> of a vector bundle $\pi : E \to X$ is a continuous map $s : X \to E$ such that
$$s(x) \in \pi^{-1}(x) = E_x$$
This is equivalent to saying $\pi(s(x)) = x$
or $\pi \circ s = id_X$

diagram
$$\begin{tikzcd} E \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd} \qquad \pi s = id_X$$

A section is C^∞ if $s : X \to E$ is a C^∞ map

<u>Observe</u>: let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, $U \subset X$ the local trivializations whose existence is provided for in the definition of a vector bundle.
Let $s : X \to E$ be a section
Then $s|_U$ maps U into $\pi^{-1}(U)$:
$$U \xrightarrow{s|_U} \pi^{-1}(U) \xrightarrow{\phi} U \times \mathbb{R}^k \xrightarrow{pr_2} \mathbb{R}^k$$

So $pr_2 \circ \phi \circ s|_U : U \to \mathbb{R}^k$ is a vector-valued function on U. Indeed, sections of a vector bundle become vector valued functions with respect to a local trivialization!

Terminology: X = base space E = total space
π = structure map of the vector bundle.
$E_x = \pi^{-1}(x)$ = fiber over $x \in X$
$(+_x, \cdot_x)$ = vector space operations on the fiber E_x
aka. fiber wise addition and scalar multiplication

<u>Examples</u>: X a smooth manifold

<u>Tangent bundle</u> $TX = \{(x,v) \mid x \in X,\ v \in T_x X\}$

The structure map is

$$\pi : TX \to X$$
$$(x,v) \mapsto x$$

Local trivializations: for $x \in X$ choose chart

U open in X, $U \ni x$, V open in $\mathbb{R}^n$

$\varphi : U \to V$ local coords

Then $D\varphi : TU \longrightarrow TV \cong V \times \mathbb{R}^n$

We identify $TU = \pi^{-1}(U)$ and $V \times \mathbb{R}^n \xrightarrow{\varphi^{-1} \times id} U \times \mathbb{R}^n$

so $\pi^{-1}(U) = TU \xrightarrow{D\varphi} TV = V \times \mathbb{R}^n \xrightarrow{\varphi^{-1} \times id} U \times \mathbb{R}^n$

is a local trivialization.

- Sections of the tangent bundle = vector fields

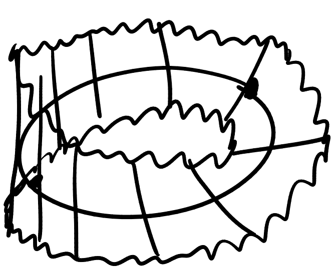
<u>Trivial bundle.</u>

$$\begin{array}{c} X \times \mathbb{R}^k \\ \downarrow pr_1 \\ X \end{array}$$

One local trivialization $U = X$!

Sections = maps $X \to \mathbb{R}^k$ or usual vector valued functions

Möbius bundle: let $X = S^1$ and let E be the rank 1 bundle shown here.



<u>Fact</u>: any section of the möbius bundle must vanish at some point $x \in S^1$. So it cannot be trivialized by a single chart.