DIFFERENTIABLE MANIFOLDS II: HOMEWORK 9

JAMES PASCALEFF

(1) Let $E \to M$ be a vector bundle with connection ∇. Let $\{e_i\}_{i=1}^r$ be a local frame for E, and let A be the matrix for ∇ in this frame. Let $\{\varepsilon^i\}_{i=1}^r$ be the frame for $E^\vee$ dual to $\{e_i\}$.
  (a) Find the matrix of the induced connection $\nabla^\vee$ on $E^\vee$ with respect the frame $\{\varepsilon^i\}$ in terms of A.
  (b) Find the matrix of the induced connection $\nabla^{\mathcal{E}nd(E)}$ on $\mathcal{E}nd(E) = E^\vee \otimes E$ with respect to the frame $\{\varepsilon^i \otimes e_j\}_{i,j=1}^r$ in terms of A.

(2) (a) Prove the differential Bianchi identity

$$dF = F \wedge A - A \wedge F$$

where A and F are the matrix-valued forms representing the connection and the curvature, respectively, with respect to some local frame. *Hint:* Use the structure equation.
  (b) Prove the differential Bianchi identity in the form

$$d_\nabla F_\nabla = 0$$

Hint: Use the formula $(d_\nabla)^2 = F_\nabla$, and compute $(d_\nabla)^3$ in two ways. Note that the symbol d_∇ appearing in the Bianchi identity is actually $d_{\nabla^{\mathcal{E}nd(E)}}$.
  (c) Work out the action of d_∇ on sections of $\Omega^\bullet(M, \mathcal{E}nd(E))$ in terms of a local frame, and show that the two forms of the differential Bianchi identity are equivalent.

(3) The exercise is about twisted de Rham cohomology of $M = S^1$ with coefficients in various flat vector bundles.
  (a) If you've never done it before, calculate the de Rham cohomology of S^1. (The twisted de Rham cohomology with coefficients in the trivial bundle $\underline{\mathbb{R}}$ with the trivial connection d).
  (b) Let $f : S^1 \to \mathbb{R}$ be a function. Verify that $\nabla = d + df$ defines a flat connection in the trivial bundle $\underline{\mathbb{R}}$. Calculate the twisted de Rham cohomology with coefficients in this flat vector bundle. *Hint:* This is straightforward if you know something about first order ordinary differential equations as in Math 285.
  (c) Writing $S^1 = \{(\cos\theta, \sin\theta \mid \theta \in [0, 2\pi)\}$, let $d\theta \in \Omega^1(S^1)$ be the angular 1-form on S^1. Note that, despite the notation, $d\theta$ is not the derivative of a function on S^1. Verify that $\nabla = d + d\theta$ defines a flat connection in the trivial bundle $\underline{\mathbb{R}}$. Calculate the twisted de Rham cohomology with coefficients in this flat vector bundle.
  (d) Let $\mu : E \to S^1$ be the Möbius bundle. Calculate the twisted de Rham cohomology with coefficients in E for any flat connection of your choosing.

(4) Let $\pi : E \to M$ be a vector bundle with a *flat* connection ∇. Let $b \in M$ be a base point. Show that the holonomy of ∇ around loops in M based at b defines an action of the fundamental group $\pi_1(M, b)$ on the fiber E_b. Conversely, show that if $\pi_1(M, b)$ acts on a vector space V, then there is a vector bundle with flat connection (E, ∇) over M and an isomorphism $V \cong E_b$ such that holonomy action on E_b corresponds to the given action on V.

(5) Let E and F be complex vector bundles over a manifold M. Using the definition in terms of curvature, Show that the total Chern class c and the Chern character ch satisfy the properties

$$c(E \oplus F) = c(E)c(F)$$

$$ch(E \oplus F) = ch(E) + ch(F)$$
$$ch(E \otimes F) = ch(E)ch(F)$$

Hint: For the two formulas involving the direct sum, show that connections ∇^E and ∇^F induce a connection on $E \oplus F$, such that the curvature matrices are "block-diagonal", that is, with respect to the isomorphism

$$\mathcal{E}nd(E \oplus F) \cong \mathcal{E}nd(E) \oplus \mathcal{H}om(F, E) \oplus \mathcal{H}om(E, F) \oplus \mathcal{E}nd(F)$$

the curvature has no components in the middle two factors.

For the formula involving the tensor product, think about what the curvature of the induced connection looks like with respect to the isomorphism

$$\mathcal{E}nd(E \otimes F) \cong \mathcal{E}nd(E) \otimes \mathcal{E}nd(F)$$

(6) Let $M = \mathbb{R}^3 \setminus \{(0, 0, z) \mid z \geq 0\}$ be the standard 3-space minus the non-negative z-axis. Let $E = M \times \mathbb{C}$ be the trivial complex line bundle over M. Define a connection on E by $\nabla = d + A$, where A is the (pure-imaginary) one-form

$$A = \alpha \frac{i}{2} \frac{1}{r(z - r)} (x\, dy - y\, dx)$$

Here r is the distance to the origin, and α is a constant. (Note that A has singularities exactly along the non-negative z-axis that was deleted.) Let $\tilde{M} = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ be the complement of the origin; $\tilde{M}$ contains M as an open submanifold.

(a) Calculate the curvature two-form F for $\nabla = d + A$ on M, and show that F extends to non-singular two-form $\tilde{F}$ on $\tilde{M}$.

(b) Calculate the integral of $\tilde{F}$ over the two sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \tilde{M}$.

(c) Show that if α is an integer, then there is a complex line bundle $\tilde{E}$ over $\tilde{M}$ with connection $\tilde{\nabla}$, such that the restriction of $(\tilde{E}, \tilde{\nabla})$ to M is isomorphic to $(E, d + A)$, and show that the curvature of this connection is precisely $\tilde{F}$.

(d) Show that the first Chern class $c_1(E) \in H^2(\tilde{M}, \mathbb{R})$ is αh, where $h \in H^2(\tilde{M}, \mathbb{Z})$ is the class satisfying $\langle h, [S^2] \rangle = 1$.

Aside: In the physics of electro-magnetism, the one-form A corresponds to the magnetic vector potential, and the two-form F to the magnetic field itself. The set-up in this problem represents a magnetic monopole with magnetic charge α. It was first considered by P. Dirac in different language.