DIFFERENTIABLE MANIFOLDS II: HOMEWORK 8

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(1) Let (M, g) be a Riemannian manifold such the for any two points $p, q \in M$, the parallel transport $T_pM \to T_qM$ is the same for all paths joining p to q. Show that M must be flat, that is, the Riemann curvature tensor is identically zero. *Hint:* Use the relationship

$$\frac{D}{\partial s}\frac{D}{\partial t}V - \frac{D}{\partial t}\frac{D}{\partial s}V = R\left(\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right)V$$

for an parametrized surface $\alpha(s, t)$ and a vector field V along the parametrized surface.

(2) Let ∇ be a connection on a vector bundle $E \to M$. Let $\{s_i\}_{i=1}^r$ and $\{\tilde{s}_i\}_{i=1}^r$ be two local frames for E (over some open set U in M). Let $G : U \to \mathrm{GL}(n, \mathbb{R})$ be the change-of-basis matrix. Let A and $\tilde{A}$ be the matrix-valued one-forms representing ∇ in the two frames, and let F and $\tilde{F}$ be the matrix-valued two-forms representing the curvature of ∇ in the two frames. Show that the transformation laws

$$\tilde{A} = G^{-1}AG + G^{-1}dG, \quad \tilde{F} = G^{-1}FG$$

are consistent with the structure equations

$$F = dA + [A, A], \quad \tilde{F} = d\tilde{A} + [\tilde{A}, \tilde{A}]$$

Recall that $[A, A]$ means the two form satisfying $[A, A](X, Y) = [A(X), A(Y)]$.

(3) Let $\mathfrak{g}$ be a Lie algebra. Recall that this means that $\mathfrak{g}$ is a vector space endowed with bilinear operation $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying skew-symmetry and the Jacobi identity:

$$[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Now let M be a manifold. There are vector bundles $\wedge^p(T^*M) \otimes \mathfrak{g}$ over M, consisting of p-forms with values in $\mathfrak{g}$. We write $\Omega^p(M, \mathfrak{g})$ for the space of sections of this bundle. The purpose of this exercise is to show that the space of $\mathfrak{g}$-valued differential forms

$$\Omega^\bullet(M, \mathfrak{g}) = \bigoplus_{p=0}^{\dim M} \Omega^p(M, \mathfrak{g})$$

is a *differential graded Lie algebra*. The exterior derivative d extends to $\mathfrak{g}$-valued forms: just differentiate the differential form part treating the $\mathfrak{g}$-coefficient as a constant. There is also an extension of the Lie bracket on $\mathfrak{g}$ to $\mathfrak{g}$-valued forms: given $\omega \in \Omega^p(M, \mathfrak{g})$ and $\eta \in \Omega^q(M, \mathfrak{g})$, first take wedge product $\omega \wedge \eta \in \Omega^{p+q}(M, \mathfrak{g} \otimes \mathfrak{g})$. The coefficients are multiplied "formally," so the result is a $\mathfrak{g} \otimes \mathfrak{g}$-valued form. Then we apply the bracket map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ to the coefficients, and the result is again a $\mathfrak{g}$-valued form that we denote

$$[\omega \wedge \eta] \in \Omega^{p+q}(M, \mathfrak{g})$$

The axioms of a differential grade Lie algebra are, first, that d is a derivation of $[\wedge]$:

$$d[\omega \wedge \eta] = [d\omega \wedge \eta] + (-1)^{\deg(\omega)}[\omega \wedge d\eta],$$

second, that $[\wedge]$ is skew-symmetric *in the graded sense*:

$$[\omega \wedge \eta] = -(-1)^{\deg(\omega)\deg(\eta)}[\eta \wedge \omega],$$

and third, that the *graded* Jacobi identity holds:

$$(-1)^{\spadesuit}[\omega \wedge [\eta \wedge \theta]] + (-1)^{\heartsuit}[\eta \wedge [\theta \wedge \omega]] + (-1)^{\clubsuit}[\theta \wedge [\omega \wedge \eta]] = 0$$

where $(-1)^{\spadesuit}, (-1)^{\heartsuit}, (-1)^{\clubsuit}$ are some signs that I leave to you to figure out.

(a) Show that $\Omega^\bullet(M, \mathfrak{g})$ is indeed a differential graded Lie algebra (figure out $\spadesuit, \heartsuit, \clubsuit$ ought to be in order to make this true).

(b) Show that, if $\mathfrak{g}$ is the Lie algebra of all $n \times n$ matrices, and A is the $\mathfrak{g}$-valued 1-form corresponding to a connection ∇, and F is the $\mathfrak{g}$-valued curvature 2-form, then the structure equation reads

$$F = dA + \frac{1}{2}[A \wedge A]$$