DIFFERENTIABLE MANIFOLDS II: HOMEWORK 7

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These problems are drawn from Chapter 9 of *A Comprehensive Introduction to Differential Geometry, Volume I,* by M. Spivak.

(1) *Definition:* if $\gamma : [a,b] \to M$ is a path, and $p : [\alpha,\beta] \to [a,b]$ is a diffeomorphism, then $c = \gamma \circ p : [\alpha,\beta] \to M$ is a *reparametrization* of γ.
  (a) Show that the length of a path is independent of its parametrization, but that the energy is not. Namely show that $\ell(c) = \ell(\gamma)$, but that $E(c) \neq E(\gamma)$ in general.
  (b) Let $\gamma : [a,b] \to M$ be a path. Show that
  $$[\ell(\gamma)]^2 \leq (b-a)E(\gamma)$$
  with equality if and only if γ has constant speed. Conclude that minimizers of energy also minimize length, and minimizers of length are reparametrizations of minimizers of energy. *Hint:* Use the Schwarz inequality
  $$\left(\int fg\right)^2 \leq \left(\int f^2\right)\left(\int g^2\right)$$
  (c) Let γ be a geodesic, and let $c = \gamma \circ p$ be a reparametrization of γ. Show that, in local coordinates, c satisfies the equation
  $$\frac{d^2c^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k(c(t))\frac{dc^i}{dt}\frac{dc^j}{dt} = \frac{dc^k}{dt}\frac{p''(t)}{p'(t)},$$
  and conversely, if c satisfies this equation then γ must be a geodesic.
  (d) Show also that if c is any path satisfying
  $$\frac{d^2c^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k(c(t))\frac{dc^i}{dt}\frac{dc^j}{dt} = \frac{dc^k}{dt}\mu(t),$$
  for some function $\mu : \mathbb{R} \to \mathbb{R}$, then c is a reparametrization of a geodesic. (This equation says that the covariant acceleration of c is proportional to the velocity).

(2) Let $\mathbb{H} = \{(x,y) \mid y > 0\}$ be the upper half plane with the Riemannian metric
$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2}$$
(The Christoffel symbols of this manifold were calculated in an earlier problem.) Show that the geodesics in $(\mathbb{H}, g)$ are precisely the semi-circles with center on the x-axis, and straight lines parallel to the y-axis, with suitable parametrizations in each case. Find the parametrizations that make those curves geodesics. *Hint:* The last part of the previous problem lets you approach this question starting with *any* parametrization of the semi-circle or line.

(3) Let (M,g) be a Riemannian manifold with the property that any two points can be joined by a unique geodesic of minimal length. Does it follow that (M,g) is complete?

(4) Let p be a point in a non-compact complete Riemannian manifold (M,g). Show that M contains an infinite geodesic "ray" starting at p. That is, find a geodesic $\gamma : [0,\infty) \to M$ with $\gamma(0) = p$ such that γ is a minimal geodesic between any two of its points.