DIFFERENTIABLE MANIFOLDS II: HOMEWORK 6

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Unless other wise specified, (M,g) is a Riemannian manifold and ∇ is its Levi-Civita connection.

(1) Let $\gamma : [a,b] \to M$ be a path.

(a) Show that the covariant derivative $\frac{D}{dt}$ along γ has the property that it is compatible with the metric: For any vector fields $V(t), W(t)$ along γ, we have

$$\frac{d}{dt} g(V(t), W(t)) = g\left(\frac{DV}{dt}, W(t)\right) + g\left(V(t), \frac{DW}{dt}\right)$$

Hint: Use the fact that the connection ∇ from which $\frac{D}{dt}$ is derived is compatible with the metric. The most direct/painful route is to use the expression in terms of Christoffel symbols. The fact that ∇ is compatible with g becomes the identity

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_\ell g_{\ell j}\Gamma^\ell_{ki} + g_{i\ell}\Gamma^\ell_{kj}$$

(b) Let $P_\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ be the parallel transport operator of the Levi-Civita connection that is defined by solving the initial value problem

$$\frac{DV}{dt} = 0, \ V(a) = V_0 \in T_{\gamma(a)}M$$

Show that P_γ preserves g in the sense that

$$g_{\gamma(a)}(v,w) = g_{\gamma(b)}(P_\gamma(v), P_\gamma(w))$$

for any $v, w \in T_{\gamma(a)}M$.

(2) Let $\alpha : [0,1] \times [0,1] \to M$, $(s,t) \mapsto \alpha(s,t)$ be a smooth parametrized surface in M.

Show that when ∇ is a torsion free connection in the tangent bundle TM,

$$\frac{D}{\partial s}\frac{\partial \alpha}{\partial t} = \frac{D}{\partial t}\frac{\partial \alpha}{\partial s}.$$

Hint: Express everything in local coordinates using Christoffel symbols. Use the fact that torsion freeness is equivalent to the symmetry condition

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Recall definitions: A vector field along α is a smooth assignment of a vector $V(s,t) \in T_{\alpha(s,t)}M$ for each (s,t) in the domain (or more precisely a smooth section of pullback vector bundle $\alpha^*TM \to [0,1] \times [0,1]$). We define a covariant partial derivative

$$\frac{DV}{\partial s}$$

as follows: for each fixed value $t = t_0$, we obtain a vector field $V(s,t_0)$ along a path $\alpha(s,t_0)$, where s is allowed to vary, and we take the covariant derivative of this vector field along this path. The other covariant partial derivative $\frac{DV}{\partial t}$ is defined analogously.

(3) Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x} \cdot \mathbf{x} = 1\}$ denote the unit sphere. Define the *round metric* g to be the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$; that is, we regard $T_\mathbf{x}S^n$ as a subspace of $T_\mathbf{x}\mathbb{R}^{n+1}$, and we let g be the dot product in the latter space.

(a) Express the round metric g in local coordinate charts on S^n. You may use whatever charts you like.

(b) Show that any orthogonal transformation $A \in \mathrm{O}(n+1)$ restricts to an isometry of (S^n, g). *Hint:* This should be obvious. *Recall definition:* An orthogonal transformation of $\mathbb{R}^{n+1}$ is a linear transformation that preserves the dot products of vectors.

(c) Show that the geodesics in (S^n, g) are the great circles (the intersection of S^n with a plane in $\mathbb{R}^{n+1}$ passing through 0). *Hint:* It suffices to show that any particular great circle is a geodesic. Then we argue that an isometry takes a geodesic to a geodesic, and that any great circle may be taken to any other great circle by an orthogonal transformation $A \in \mathrm{O}(n+1)$.

(d) Calculate the parallel transport of the Levi-Civita connection along a geodesic/great circle. That is let γ be a segment of a great circle connecting points p and q, and find the operator

$$P_\gamma : T_p S^n \to T_q S^n$$

Hint: The answer is related to the rotation of the sphere in the plane of the great circle containing γ.

(4) Let $U : M \to \mathbb{R}$ be a smooth function consider the functional defined on smooth paths $\gamma : [a, b] \to M$ called the *action*

$$S(\gamma) = \int_a^b \left[\frac{1}{2} g(\dot\gamma(t), \dot\gamma(t)) - U(\gamma(t)) \right] dt$$

Fix two points p and q in M, and consider the restriction of this functional to the set of paths such that $\gamma(a) = p$ and $\gamma(b) = q$. Show that a critical point of the restricted functional must satisfy the equation

$$\frac{D}{dt}(\dot\gamma) = -\operatorname{grad} U.$$

Definition: The gradient of a function U on a Riemannian manifold (M, g) is the unique vector field $\operatorname{grad} U$ such that

$$g(v, \operatorname{grad} U) = dU(v),$$

where dU is the differential of U and v is a tangent vector.