DIFFERENTIABLE MANIFOLDS II: HOMEWORK 5

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Throughout, let (M, g) denote a connected Riemannian manifold.

DEFINITION: an **isometry** between two Riemannian manifolds (M, g) and (N, h) is a diffeomorphism $f : M \to N$ with the property that the inner product with respect to g of two tangent vectors to M equals the inner product with respect to h of the corresponding vectors in N. That is to say

$$g_p(v, w) = h_{f(p)}(Df_p(v), Df_p(w))$$

for all $p \in M$ and all $v, w \in T_pM$. Two Riemannian manifolds are **isometric** if there is some isometry between them.

(1) Let $U \subset \mathbb{R}^n$ be an open set. Let $v \cdot w$ denote the ordinary dot product of tangent vectors $v, w \in T_pU$. Let g be a Riemannian metric on U (different from dot product). Show that, for any compact subset K in U, there are positive constants $C_1, C_2 > 0$ such that the inequalities

$$C_1(v \cdot v) \le g_p(v, v) \le C_2(v \cdot v)$$

hold for all $p \in K$ and all $v \in T_pU$. (The constants C_1, C_2 may depend on the choice of compact subset K).

Furthermore, formulate and prove an analogous inequality for the lengths of paths in K.

Hint: Regard $(p, v) \mapsto g_p(v, v)$ as a function on the set of unit vectors with respect to ordinary dot product in tangent bundle of U,

$$T_1U = \{(p, v) \mid p \in U,\ v \in T_pU,\ v \cdot v = 1\},$$

and use the fact that a continuous positive function on a compact set is bounded away from 0 and ∞.

Alternative Hint: In terms of the coordinates $(x^1, \dots, x^n)$ in $U \subset \mathbb{R}^n$, the metric g has the representation

$$g_p = \sum_{i,j=1}^{n} g_{ij}(p)\, dx^i \otimes dx^j.$$

The matrix $G(p) = (g_{ij}(p))_{i,j=1}^n$ is a positive-definite symmetric matrix (in the sense of "ordinary" linear algebra) at each point $p \in U$. Show that we can take $C_1 = \lambda_{min}$ and $C_2 = \lambda_{max}$, where λ_{min} is the minimal eigenvalue of $G(p)$ as p ranges over the compact set K, and λ_{max} is similarly the maximal eigenvalue of $G(p)$.

(2) Let $\mathrm{dist}_g : M \times M \to \mathbb{R}$ be the Riemannian distance function. Complete the proof that dist_g defines a metric space structure by showing that $\mathrm{dist}_g(p, q) = 0 \implies p = q$. *Hint:* use Problem 1.

(3) Define the open g-ball with center $p \in M$ and radius $r > 0$ as

$$B_g(p, r) = \{q \in M \mid \mathrm{dist}_g(p, q) < r\}.$$

Since dist_g is a metric, these sets form a basis for a topology: Call a set g-open if it is equal to a union of open g-balls. Show that the notion of g-open sets is redundant: a set is g-open iff it is open in the original underlying topology of M.

Hint: Reduce to showing that each open g-ball is open, and that each open set contains an open g-ball around each of its points. For these statements, use local coordinates and Problem 1.

(4) Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ denote the upper half plane. We give this space the metric

$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2}$$

Calculate the Levi-Civita connection for this Riemannian manifold.

(5) Let g be a Riemannian metric on $\mathbb{R}$. Calculate the Christoffel symbol of the Levi-Civita connection of $(\mathbb{R}, g)$. Prove that $(\mathbb{R}, g)$ is isometric to an interval $\subset \mathbb{R}$ equipped with the standard Euclidean metric. In fact we can take I to be either $(0, L)$ for some $L > 0$, $(0, \infty)$, or $\mathbb{R}$ itself. *Hint:* Write down a differential equation that the isometry must satisfy.

(6) Now let g be a Riemannian metric on an open set $U \subset \mathbb{R}^2$. We ask, is (U, g) isometric to any open set in the standard Euclidean space $\mathbb{R}^2$? Show that the resulting system of partial differential equations for the isometry is overdetermined (there are more equations than unknown functions). This suggests already that the general answer is "no," because this system of PDE could be inconsistent. Find a particular case where the system is inconsistent. (A system of partial differential equations is inconsistent if the equations imply $0 = 1$. An example is the system

$$\frac{\partial f}{\partial x} = 0, \; \frac{\partial f}{\partial y} = x$$

for a function $f(x, y)$. Compare the y derivative of the first equation with the x derivative of the second equation.)