DIFFERENTIABLE MANIFOLDS II: HOMEWORK 4

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(1) Let H be a horizontal distribution on a smooth fiber bundle $\pi : E \to M$ with typical fiber F. Assume that parallel transport always exists for this connection. Recall the holonomy group

$$\mathrm{Hol}_x(H) \subset \mathrm{Diff}(E_x)$$

consisting of all parallel transport maps of all piecewise smooth loops based at x.

(a) Since E_x is diffeomorphic to F, $\mathrm{Hol}_x(H)$ may be regarded as a subgroup of $\mathrm{Diff}(F)$. This subgroup depends on

(i) The choice of the base point $x \in M$, and

(ii) The choice of diffeomorphism $E_x \cong F$.

Show that if M is connected then $\mathrm{Hol}_x(H)$ is actually independent of these choices up to conjugation in $\mathrm{Diff}(F)$. (That is to say, if we modify either of the choices then the subgroup changes by conjugation by some $g \in \mathrm{Diff}(F)$.)

(b) Suppose that M is connected, and that there is some $x \in M$ such that the holonomy group is trivial: $\mathrm{Hol}_x(H) = \{\mathrm{id}\}$. Show that the fiber bundle $\pi : E \to M$ is trivial. That is, show there is a commutative diagram of smooth maps

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "\pi"'] & & M \times F \arrow[dl, "\mathrm{pr}_1"] \\ & M & \end{tikzcd}$$

(2) Explicitly construct an example of a fiber bundle with connection such that the holonomy group is not trivial, even though the underlying bundle is smoothly trivial. For (moral) bonus points, try to make the holonomy group as large as you can. (This exercise is intentionally open-ended.)

(3) (a) Prove the universal property of the tensor product $V \otimes W$ from the definition. (See pages 1–2 of lecture 8.)

(b) Prove that there is a canonical isomorphism $(V \otimes W)^\vee \cong V^\vee \otimes W^\vee$. Hint: an element of either side can be contracted with an element of $V \otimes W$, and the isomorphism should respect this.

(4) Let ∇ be a covariant derivative in a vector bundle $\pi : E \to M$, and let $H_\nabla \subset TE$ be the associated horizontal distribution. Let $\gamma : [a, b] \to M$ be a path. Prove that the parallel transport map for H_∇

$$P_\gamma : \pi^{-1}(\gamma(a)) \to \pi^{-1}(\gamma(b))$$

is a *linear* isomorphism between vector spaces, not merely a diffeomorphism between manifolds. You should use the characterization of horizontal lifting in terms of vanishing of the covariant derivative.

(5) Let $\pi : E \to M$ be a vector bundle. Let ∇ be a covariant derivative in E. There is a horizontal distribution H_∇ in E corresponding to ∇. Recall from lecture 7 that a horizontal distribution always corresponds to a section

$$\omega \in \Gamma(E, T^*E \otimes (TE)^v)$$

such that for a vertical vector $X \in (TE)^v$, we have

$$\omega(X) = X$$

(Here $\omega(X)$ denotes the contraction of an element of $T^*E \otimes (TE)^v$ with an element of TE, resulting in an element of $(TE)^v$.) The horizontal distribution is then recovered as the set of vectors X such that $\omega(X) = 0$.

Figure out the correspondence $\nabla \to \omega$. First do this with respect to local coordinates on M and a local trivialization of E. Hint: ω may be expressed in terms of Christoffel symbols. Then check that the result does not depend on the choice of coordinates or trivialization. (See also the proof that H_∇ is a subbundle contained in the notes for lecture 9.)