DIFFERENTIABLE MANIFOLDS II: HOMEWORK 3

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(1) Tensoriality: Let $E_1, \ldots, E_k, F$ be smooth vector bundles over a smooth manifold M. Any section ω of $E_1^\vee \otimes \cdots \otimes E_k^\vee \otimes F$ determines a map

$$\Omega : \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_k) \to \Gamma(M, F),$$

where $\Omega(s_1, \ldots, s_k)$ is defined by evaluating the $E_i^\vee$ component of ω on s_i. Any function Ω so obtained is *tensorial*, meaning that it is $C^\infty(M)$-linear in each input. Prove that the correspondence between such sections ω and such tensorial functions Ω is a bijection.

(2) A connection $\nabla : \mathfrak{X}(M) \otimes \Gamma(M, E) \to \Gamma(M, E)$ is, by definition, tensorial ($C^\infty(M)$-linear) in the vector field input $X \in \mathfrak{X}(M) = \Gamma(M, TM)$. Show that, for $x_0 \in M$, the covariant derivative $(\nabla_X s)(x_0)$ depends only on $X(x_0)$. Show that the connection may be reinterpreted as a map

$$\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$$

When interpreted this way, show that the Leibniz rule becomes

$$\nabla(fs) = df \otimes s + f\nabla s$$

(3) Show that every vector bundle over a (second-countable Hausdorff) manifold admits a connection. The idea is to use local trivializations to get local connections, and then use a partition of unity to interpolate between them. Beware that the sum of two connections is not a connection!

(4) Consider the n-th order ordinary differential equation

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

where t is the independent variable, $y(t)$ is a real-valued function, and f is a smooth function of all arguments. There is a standard trick that reduces a n-th order equation to a first-order system of n equations: introduce new independent variables y_i corresponding to $y^{(i)}$ and impose the equations $y_i = y'_{i-1}$.

Using this trick, show that the ODE equation corresponds to a connection in the sense of horizontal distribution on the bundle $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$; the base $\mathbb{R}$ factor corresponds to t and the fiber $\mathbb{R}^n$ corresponds to $(y, y', \ldots, y^{(n-1)})$. Again, solutions should correspond to horizontal sections.

For the second part, suppose that the original ODE is linear in the conventional sense:

$$y^{(n)} = p_0(t)y + p_1(t)y' + \cdots + p_{n-1}(t)y^{(n-1)}$$

Show that this equation also corresponds to a connection *in the sense of covariant derivative* in the *vector bundle* $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. That is, find a covariant derivative ∇ on this bundle such that solutions of the ODE correspond to solutions of

$$\nabla_{\frac{\partial}{\partial t}} s = 0$$