DIFFERENTIABLE MANIFOLDS II: HOMEWORK 2

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(1) Find two vector bundles over the same base manifold X which are isomorphic as vector bundles in the general sense, but not isomorphic *over* X. That is to say, there is no isomorphism between the bundles covering the identity map on X.

(2) Prove the statement that if two smooth maps $f : X \to Z$ and $g : Y \to Z$ are transverse, then the fiber product $X \times_Z Y$ is a smooth submanifold of $X \times Y$. (This is an application of the implicit function theorem.)

(3) Using the universal property of the fiber product, show that any morphism of bundles over possibly different bases

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_1 \\ \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

gives rise to a morphism *over* X_1

$$\begin{array}{ccc} E_1 & \longrightarrow & f^*E_2 \\ & {\scriptstyle \xi_1}\searrow \quad \swarrow{\scriptstyle f^*\xi_2} & \\ & X_1 & \end{array}$$

Show that the original morphism is a bundle map (= fiberwise isomorphism) iff the latter morphism is an isomorphism.

(4) Let $\xi : E \to X$ be a rank r vector bundle. For an integer k, $1 \leq k \leq r$, a *smooth k-frame* in E is a collection of k smooth sections $s_1, \ldots, s_k$ of ξ such that, for every $x \in X$, the vectors $s_1(x), \ldots, s_k(x) \in E_x$ are linearly independent. Show that a vector bundle is trivial if and only if it admits an r-frame (r = rank).

(5) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and suppose that 0 is a regular value, so that $X = f^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^n$. If $i : X \to \mathbb{R}^n$ denotes the inclusion map, show that there is an isomorphism of bundles over X:

$$TX \oplus \underline{\mathbb{R}} \cong i^*T\mathbb{R}^n$$

(here TX denotes the tangent bundle of X, $\underline{\mathbb{R}}$ denotes the trivial rank one bundle over X, $\oplus$ denotes the Whitney sum, and $i^*T\mathbb{R}^n$ is the pullback of the tangent bundle of the ambient $\mathbb{R}^n$).

Formulate and prove the analogous statement for an $(n-k)$–dimensional submanifold $Y \subset \mathbb{R}^n$ which is transversely cut out by k smooth functions $f_1, \ldots, f_k$:

$$Y = \bigcap_{i=1}^{k} f_i^{-1}(0)$$

("Transversely cut out" means that 0 is a regular value of the map $\mathbf{F} = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$.)

(6) (Optional, if you enjoy categories and functors, but the application is very important) Let $\mathcal{V}$ denote the category of finite-dimensional $\mathbb{R}$-vector spaces and linear transformations, and

let $\mathcal{V}(X)$ denote the category of finite-rank vector bundles over X and morphisms over X (covering the identity map on X). For two categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the category of functors $\mathcal{C} \to \mathcal{D}$, where morphisms are natural transformations.

In the lecture we assocated to each functor

$$T : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$$

and each collection of vector bundles $\xi_i : E_i \to X$, $i = 1, \ldots, n$ a vector bundle

$$T(\xi_1, \ldots, \xi_n) : T(E_1, \ldots, E_n) \to X$$

(a) Show that this construction is compatible with morphisms in $\mathcal{V}(X)$. That is, show that it defines a functor

$$T(X) : \mathcal{V}(X) \times \cdots \times \mathcal{V}(X) \to \mathcal{V}(X).$$

(b) Show that this construction is compatible with natural transformations. That is show that it defines a functor

$$\mathrm{Fun}(\mathcal{V}^{\times n}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{V}(X)^{\times n}, \mathcal{V}(X)), \quad T \mapsto T(X).$$

(c) As an application, deduce that for any vector bundles E, F over X, there are natural morphisms over X:

$$E^\vee \otimes F \to \mathcal{H}om(E, F)$$
$$\mathcal{H}om(E, F) \otimes E \to F$$
$$E \to (E^\vee)^\vee$$
$$E^\vee \otimes E \to \underline{\mathbb{R}}$$

(7) A *exact sequence* of vector bundles over X is a sequence of morphisms over X

$$\cdots \to E \to F \to G \to \cdots$$

such that for all $x \in X$, the sequence

$$\cdots \to E_x \to F_x \to G_x \to \cdots$$

on fibers over x is an exact sequence of vector spaces. Let 0 denote the rank 0 vector bundle $\mathrm{id} : X \to X$.

(a) Show that a fiberwise injective morphism $0 \to E' \to E$ may be completed to a short exact sequence of vector bundles over X

$$0 \to E' \to E \to E'' \to 0$$

(b) Show that a fiberwise surjective morphism $E \to E'' \to 0$ may be completed to a short exact sequence of vector bundles over X.

$$0 \to E' \to E \to E'' \to 0$$

(c) Show by a counterexample that not every morphism over X has a kernel, where a *kernel* for a morphism $E \to F$ is a vector bundle K fitting into an exact sequence

$$0 \to K \to E \to F$$

(d) Show by a counterexample that not every morphism over X has a cokernel, where a *cokernel* for a morphism $E \to F$ is a vector bundle C fitting into an exact sequence

$$E \to F \to C \to 0$$